PHS 3031 QUANTUM MECHANICS — SOLUTIONS

Exercise #1 • "The magnitude of the current is proportional to the intensity of the light striking the metal" — the number of photons, which strike the metal per unit area per unit time, is proportional to the intensity of the incident light. The number of electrons ejected, per unit area per unit time (namely the current) is thus proportional to the incident light intensity. • "The speed of the ejected electron is independent of the intensity of the light striking the metal" — the intensity of the light affects the number of photons, but not the physical form of each photon. Therefore, under the photon hypothesis, the intensity of the incident light has no effect upon the speed of an electron (which is ejected under the influence of a single photon). • "The speed of the ejected electrons is dependent on the frequency of the incident ultraviolet light": see equation (2) of the notes! • "The rate, at which electrons are emitted, is proportional to the intensity of the incident UV light". The number of photons, which strike the metal per unit area per unit time, is proportional to the intensity of the incident radiation. The rate at which photoelectrons are emitted is proportional to the rate at which photons strike the metal, a quantity which is itself proportional to the intensity. • "Ejected electrons are observed as soon as the UV lamp is turned on." The intensity of the lamp, even when very low (but strictly non zero), has essentially no influence on the fact that photoelectrons are observed as soon as the lamp is turned on (the finite speed of light must be allowed for, of course). A "whole" photon may be absorbed, leading to electron emission, as soon as the lamp is on — low intensity just implies that fewer such photoelectrons are emitted per unit time.

Exercise #2 With reference to FIG 2, let the energy and momentum of the incident photon be respectively denoted by $E_\gamma^{(i)}$ and $\vec{p}_\gamma^{(i)}$, with the initial energy and momentum of the electron being given by $E_e^{(i)}$ and $\vec{p}_e^{(i)} = \vec{0}$. After the collision these quantities become $E_\gamma(f)$, $\vec{p}_\gamma^{(f)}$, $E_e^{(f)}$ and $\vec{p}_e^{(f)}$, respectively. Separately invoking energy and momentum conservation, we have:

(A) $E_\gamma^{(i)} + E_e^{(i)} = E_\gamma^{(f)} + E_e^{(f)}$ —— energy conservation

(B) $\vec{p}_\gamma^{(i)} = \vec{p}_\gamma^{(f)} + \vec{p}_e^{(f)}$ —— momentum conservation

To proceed further, we need to use three observations:

(i) Since the electron is initially stationary, its energy will be equal to its rest-mass energy, so that:

(C) $E_e^{(i)} = m_e c^2$, where m_e = rest mass of electron

(ii) The energy E_γ of a photon is related to its wavelength λ_γ by $E_\gamma = hc/\lambda_\gamma$, so that:

(D) $E_\gamma^{(i)} = hc/\lambda_\gamma^{(i)}$, $\quad E_\gamma^{(f)} = hc/\lambda_\gamma^{(f)}$

(iii) The energy E, momentum $\vec{p}$, and rest mass m_0, of a material particle in free space, obey the relativistic energy-momentum-mass relationship:

(E) $E^2 = m_0^2 c^4 + |\vec{p}|^2 c^2$.

For the case of the scattered electron, this gives:

(F) $E_e^{(f)} = \sqrt{m_e^2 c^4 + |\vec{p}_e^{(f)}|^2 c^2}$.

Substitute (C), (D) and (F) into (A). Isolating the square root gives:

(G) $$m_e c^2 + hc\left[\frac{1}{\lambda_\gamma^{(i)}} - \frac{1}{\lambda_\gamma^{(f)}}\right] = \sqrt{m_e^2 c^4 + |\vec{p}_e^{(f)}|^2 c^2}.$$

Squaring the above expression (which gets rid of the pesky square root!), then cancelling m_e^2 and dividing by c^2 gives:

(H) $$h^2\left[\frac{1}{\lambda_\gamma^{(i)}} - \frac{1}{\lambda_\gamma^{(f)}}\right]^2 + 2m_e hc\left[\frac{1}{\lambda_\gamma^{(i)}} - \frac{1}{\lambda_\gamma^{(f)}}\right] = |\vec{p}_e^{(f)}|^2.$$

Next, we wish to eliminate the electron recoil momentum $p_e^{(f)}$, as we are primarily interested in the properties of the Compton-scattered photon. To this end, isolate the electron recoil momentum on one side of (B), and then take the squared modulus of the result. This yields:

(I) $$|\vec{p}_e^{(f)}|^2 = |\vec{p}_\gamma^{(i)} - \vec{p}_\gamma^{(f)}|^2 = |\vec{p}_\gamma^{(i)}|^2 + |\vec{p}_\gamma^{(f)}|^2 - 2\vec{p}_\gamma^{(i)} \cdot \vec{p}_\gamma^{(f)}.$$

Regarding the first & second term on the right side of equation (I), we note the de Broglie relation $\lambda_\gamma = h/|\vec{P}_\gamma|$ for the wavelength λ_γ and momentum $\vec{P}_\gamma$ of a photon, so that:

(J) $|\vec{P}_\gamma{}^{(i)}| = h/\lambda_\gamma{}^{(i)}, \quad |\vec{P}_\gamma{}^{(f)}| = h/\lambda_\gamma{}^{(f)}$.

Further, regarding the third term on the right side of (I), we have:

(K) $$\vec{P}_\gamma{}^{(i)} \cdot \vec{P}_\gamma{}^{(f)} = |\vec{P}_\gamma{}^{(i)}|\,|\vec{P}_\gamma{}^{(f)}| \cos\theta = \frac{h^2 \cos\theta}{\lambda_\gamma{}^{(i)} \lambda_\gamma{}^{(f)}},$$

where use has been made of (J), and θ is the angle through which the incident photon has been scattered (see Fig 2).

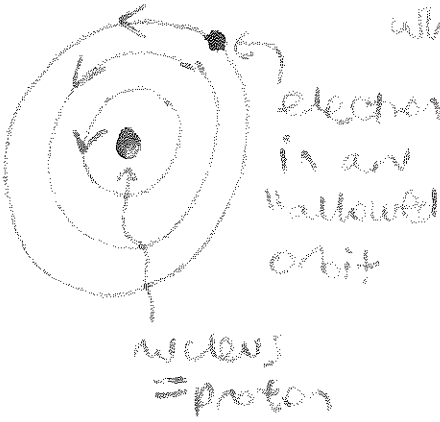
Substitute (J) and (K) into the right side of (I). The resulting formula, for the square of the recoil momentum of the electron, may then be used to eliminate this quantity from (H), to give:

(L) $$h^2\left[\frac{1}{\lambda_\gamma{}^{(i)}} - \frac{1}{\lambda_\gamma{}^{(f)}}\right]^2 + 2m_e hc\left[\frac{1}{\lambda_\gamma{}^{(i)}} - \frac{1}{\lambda_\gamma{}^{(f)}}\right] = \frac{h^2}{[\lambda_\gamma{}^{(i)}]^2} + \frac{h^2}{[\lambda_\gamma{}^{(f)}]^2} - \frac{2h^2\cos\theta}{\lambda_\gamma{}^{(i)}\lambda_\gamma{}^{(f)}}.$$

After a little algebra, together with the trigonometric formula:

(M) $\sin^2(\theta/2) = \frac{1}{2}(1-\cos\theta)$, one arrives at Compton's formula: (N) $\lambda_\gamma{}^{(f)} - \lambda_\gamma{}^{(i)} = \dfrac{2h}{m_e c}\sin^2(\theta/2)$.

<u>Exercise #3</u> Bohr's model assumes the hydrogen atom's electron to move in circular orbits, each of which are labelled by a given integer n ("principal quantum number"). In a given circular orbit, the Coulomb force of attraction between electron and proton (nucleus) is balanced against centripetal acceleration:



(A) $$\frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r}$$

v → electron speed

r → radius of orbit

m → electron mass

"e" is magnitude of charge on electron and proton

ε_0 → permittivity of free space

Now, the magnitude L of the angular momentum, associated with a given Bohr orbit, is:

(B) $L = m_e v r$.

Bohr assumed that L was quantised, postulating that it could only attain the discrete values:

(C) $L = n\hbar$, $n = 1, 2, 3, \ldots$

Combining (B) and (C), we have:

(D) $m_e v r = n\hbar$.

Considered as a pair, the simultaneous equations (A) and (D) may be solved for v and r, to give:

(E) $v = \dfrac{e^2}{4\pi\epsilon_0 \hbar n}$

(F) $r = \dfrac{4\pi\epsilon_0 \hbar^2 n^2}{m_e e^2}$

Now, denote the total energy of the electron by E_n. The "n" subscript is there to indicate that the electron energy will depend on the principal quantum number n. The total energy of the electron, in a Bohr orbit labelled by n, will be the sum of its kinetic and potential energies:

(G) E_n = Kinetic energy + potential energy

$$= \frac{1}{2} m_e v^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

taken from first-year electrostatics! ☺

(E)

$$= \frac{1}{2} m_e \left\{ \frac{e^2}{4\pi\epsilon_0 \hbar n} \right\}^2 - \frac{e^2}{4\pi\epsilon_0 \left\{ \frac{4\pi\epsilon_0 \hbar^2 n^2}{m_e e^2} \right\}}$$

from (F)

$$= \frac{-m_e e^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2}, \quad n = 1, 2, 3 \ldots$$

QED! ☺

Exercise #4 Begin with eqn (5) of the notes:

Ⓐ $E_m - E_n = E_\gamma$. In the present context, E_m is the energy of the Bohr atom with the electron in the m^{th} Bohr orbit, with E_n similarly defined. Making use of the Bohr formula:

Ⓑ $E_m = \dfrac{-m_e e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 m^2}$ & $E_n = \dfrac{-m_e e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 n^2}$,

we see that Ⓐ becomes:

Ⓒ $$-\frac{m_e e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 m^2} - -\frac{m_e e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 n^2} = \underbrace{E_\gamma}_{\text{energy of emitted photon}} = \frac{hc}{\lambda_\gamma}$$

wavelength of emitted photon

see eqn (4) of notes

Solving the above equation, for the wavelength λ_γ of the emitted photon, we obtain:

Ⓓ $$\lambda_\gamma = \frac{8h^3 c \varepsilon_0^2 n^2 m^2}{m_e e^4 (m^2 - n^2)}.$$ The $n = 2$ case of this formula is:

Ⓔ $$\lambda_\gamma = \underbrace{\frac{32 h^3 c \varepsilon_0^2}{m_e e^4}}_{\text{call this "C"}} \times \frac{m^2}{m^2 - 4} \equiv \frac{Cm^2}{m^2-4}, \quad m = \text{integer}.$$

Q.E.D.

Note that Ⓓ is more general than Ⓔ. Q.E.D.

Exercise #5 Equation (2) of the notes may be written in the modified form:

Ⓐ $E_\lambda = \dfrac{hc}{\lambda} - W$.

W → work function

λ → wavelength of illuminating photon

E_λ → energy of emitted electron, corresponding to illuminating photon of wavelength λ.

Write out the $\lambda = \lambda_1$ ("first wavelength") and $\lambda = \lambda_2$ ("second wavelength") cases of Ⓐ:

Ⓑ $$\begin{cases} E_{\lambda_1} = hc/\lambda_1 - W, \\ E_{\lambda_2} = hc/\lambda_2 - W. \end{cases}$$

Now subtract the pair of equation in (B), thereby eliminating W. Solve the resulting expression for h:

$$\text{(C)}\quad h = \frac{E_{\lambda_1} - E_{\lambda_2}}{c\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)}.$$

charge on electron

Now, we have: $E_{\lambda_1} = 2.1$ electron volts $= 2.1 \times e$ Joules
$E_{\lambda_2} = 0.5$ " " $= 0.5 \times e$ Joules
$\lambda_1 = 0.3$ micron $= 0.3 \times 10^{-6}$ metres
$\lambda_2 = 0.5$ micron $= 0.5 \times 10^{-6}$ metres

→ Note that we are converting all numerical values into SI units!

$c = 3.0 \times 10^{8}$ m/s

$e = 1.6 \times 10^{-19}$ coulombs

With the above numerical values (in SI units), (C) yields the following numerical estimate for h:

$$\text{(D)}\quad h \simeq \frac{2.1e - 0.5e}{c\left(\frac{1}{0.3\times10^{-6}} - \frac{1}{0.5\times10^{-6}}\right)} = \underline{\underline{6.4 \times 10^{-34}\ \text{Js}}}$$

This may be compared to the current value $h = 6.626... \times 10^{-34}$ Js.

<u>Exercise #6</u> Take the Compton formula, as given by eqn. (?) in the notes, and write it in the form:

$$\text{(A)}\quad \lambda' = \lambda + 2\lambda_c \sin^2\left(\frac{\theta}{2}\right)$$

→ angle through which photon is scattered

→ wavelength of incident photon

wavelength of scattered photon

$\lambda_c = \frac{h}{m_e c}$ = "Compton wavelength" $= 2.43 \times 10^{-12}$ m

~~Find the initial~~

Note: $\lambda = 10^{-12}$ m

(i) $\theta = 60^\circ \Rightarrow \lambda' = \lambda + 2\lambda_c \sin^2(\theta/2)$

$$= 10^{-12}m + 2(2{\cdot}43\times10^{-12}m)\sin^2\left(\frac{60^\circ}{2}\right)$$

$$= \underline{2{\cdot}2\times10^{-12}\,m}$$

(ii) $\theta = 90^\circ \Rightarrow \lambda' = \lambda + 2\lambda_c \sin^2\left(\frac{\theta}{2}\right)$

$$= 10^{-12}m + 2(2{\cdot}43\times10^{-12}m)\sin^2\left(\frac{90^\circ}{2}\right)$$

$$= \underline{3{\cdot}4\times10^{-12}\,m}$$

(iii) $\theta = 120^\circ \Rightarrow \lambda' = \lambda + 2\lambda_c \sin^2\left(\frac{\theta}{2}\right)$

$$= 10^{-12}m + 2(2{\cdot}43\times10^{-12}m)\sin^2\left(\frac{120^\circ}{2}\right)$$

$$= \underline{4{\cdot}6\times10^{-12}\,m}$$

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exercise # 7 From (12), a surface of constant phase obeys:

(A) $\varphi_0 = \vec{k}\cdot\vec{r} - \omega t$, where φ_0 is a real constant. Also, at a given fixed time "t", the quantity ωt (in (A)) may also be considered constant. Let:

(B) $\vec{k} \equiv (k_x, k_y, k_z)$, i.e., let k_x denote the x component of $\vec{k}$, let k_y denote the y component of $\vec{k}$, etc. Similarly,

(C) $\vec{r} \equiv (x, y, z)$. Thus (A) becomes:

(D) $0 = k_x x + k_y y + k_z z + \underline{\text{constant}}$.

this constant is equal to $\varphi_0 - \omega t$ (for a given fixed t).

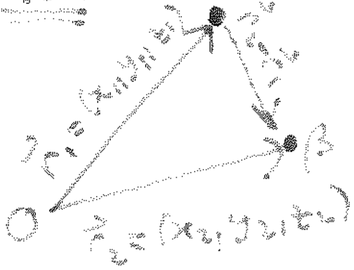
Explicitly, we have:

(E) $0 = k_x x + k_y y + k_z z + \varphi_0 - \omega t$.

(i) Equation (E) defines a plane — if this point is unclear, please consult any text on linear algebra (and/or do a search for "mathworld" on the net!).

Next, we need to show that $\vec{k}$ is normal to this plane. Again, this point will be explained in any text on linear algebra; having said that, I'll give a proof here ☺. With reference to FIG A,

FIG A



Let α and β denote points in the plane defined by equation (E). With respect to a given origin O, the position vectors of α and β are, respectively:

(F) $\alpha: \vec{r}_1 \equiv (x_1, y_1, z_1)$,

(G) $\beta: \vec{r}_2 \equiv (x_2, y_2, z_2)$.

From (E), we can obtain an expression for z_1 as a function of x_1 & y_1; similarly, we can obtain an expression for z_2 as a function of x_2 and y_2. Thus:

(H) $\alpha: \vec{r}_1 = \left(x_1, y_1, \dfrac{\omega t - \varphi_0 - k_x x_1 - k_y y_1}{k_z}\right)$

(I) $\beta: \vec{r}_2 = \left(x_2, y_2, \dfrac{\omega t - \varphi_0 - k_x x_2 - k_y y_2}{k_z}\right)$

The vector pointing from α to β, namely $\vec{r}_2 - \vec{r}_1$, is:

(J) $\vec{r}_2 - \vec{r}_1 = \left(x_2 - x_1, y_2 - y_1, \dfrac{k_x x_1 + k_y y_1 - k_x x_2 - k_y y_2}{k_z}\right)$.

This vector $(\vec{r}_2 - \vec{r}_1)$ lies in the plane of constant φ (at a given fixed t). To complete the proof that $\vec{k}$ is normal to this plane, we need merely show that the dot product of $\vec{k}$ and $\vec{r}_2 - \vec{r}_1$ is zero:

(K) $\vec{k} \cdot (\vec{r}_2 - \vec{r}_1)$

$$\overset{(J)}{=} (k_x, k_y, k_z) \cdot \left(x_2 - x_1, y_2 - y_1, \frac{1}{k_z}(k_x x_1 + k_y y_1 - k_x x_2 - k_y y_2)\right)$$

$$= k_x(x_2 - x_1) + k_y(y_2 - y_1) + k_z \frac{1}{k_z}(k_x x_1 + k_y y_1 - k_x x_2 - k_y y_2)$$

$$= 0 \quad \text{QED!}$$

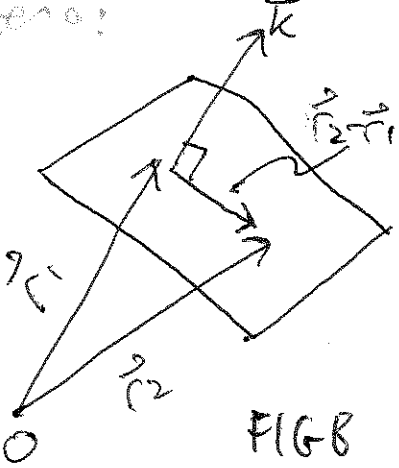


FIG 8

(iii) From (E), the surfaces of constant phase W and $W+2\pi$ are respectively given by:

$$\begin{cases} (L)\ 0 = k_x x + k_y y + k_z z + W - \omega t, \\ (M)\ 0 = k_x x + k_y y + k_z z + W + 2\pi - \omega t. \end{cases}$$

From (i), both of the above planes are perpendicular to $\vec{k}$; therefore, both of the above planes are parallel to one another (see FIG C).

In general, neither of these planes will pass through the origin "O" – see FIG C. However, if we let

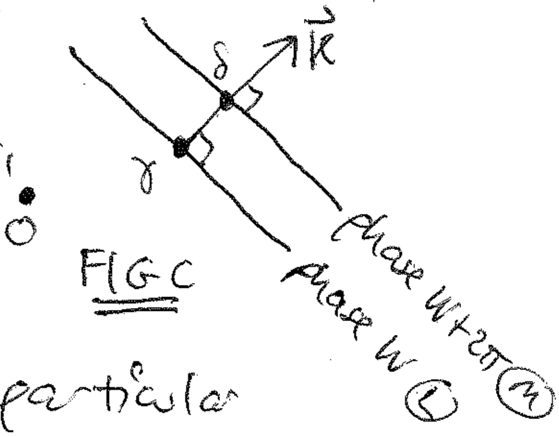


FIG C

(N) $W - \omega t = 0$,

which amounts to choosing the particular time $t = W/\omega$, then (L) & (M) reduce to:

(L') $0 = k_x x + k_y y + k_z z$,

(M') $0 = k_x x + k_y y + k_z z + 2\pi$;

the origin $(x, y, z) = (0, 0, 0)$ now lies in plane L' – see FIG D. Put the vector $\vec{k}$ with its tail at the origin, as shown. Suppose this vector to pierce the plane M' at the point F, as shown. Evidently, the vector from O to F is some multiple of $\vec{k}$, so that we may write:

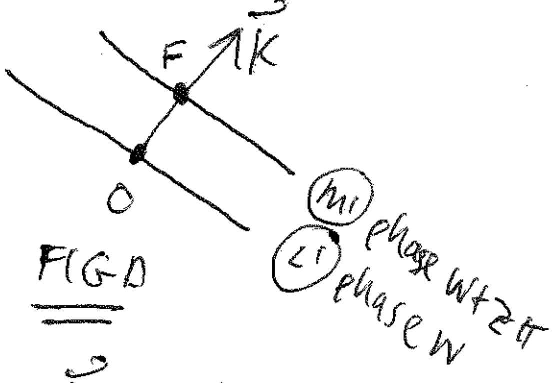


FIG D

(P) vector from O to F $\equiv \vec{OF} = \tau \vec{k}$. (some real number)

We need to choose τ such that the vector $\tau\vec{k}$ has x, y, and z components that obey equation (M').

Thus M' becomes:

(Q) $$0 = K_x[\underbrace{\tau K_x}_{x \text{ component of } \tau\vec{K}}] + K_y[\underbrace{\tau K_y}_{y \text{ component of } \tau\vec{K}}] + K_z[\underbrace{\tau K_z}_{z \text{ component of } \tau\vec{K}}] + 2\pi,$$

which is readily solved for τ to give:

(R) $$\tau = -2\pi/(K_x^2+K_y^2+K_z^2).$$

Thus (P) becomes (see also FIG D):

(S) vector from O to F $\equiv \vec{OF} = \tau\vec{K} = \dfrac{-2\pi}{K_x^2+K_y^2+K_z^2}\underbrace{(K_x,K_y,K_z)}_{\vec{K}}$.

We can now answer part (ii) of the exercise:

(T) Distance between planes (L') and (M') (see FIG D)

$= |\vec{OF}|$ → (magnitude of vector $\vec{OF}$)

$$= \left|\frac{-2\pi}{K_x^2+K_y^2+K_z^2}(K_x,K_y,K_z)\right|$$

$$= \frac{2\pi}{K_x^2+K_y^2+K_z^2}\times\sqrt{K_x^2+K_y^2+K_z^2}$$

$$= 2\pi/\sqrt{K_x^2+K_y^2+K_z^2}$$

$= 2\pi/|\vec{K}|$ ... now use eqn. (9) of notes

$= 2\pi/(2\pi/\lambda) = \lambda$ → the de Broglie wavelength. QED!

(iii) I'll leave this to you!! ☺ Hint: consider how the wave front moves as time increases from t to $t+\delta t$, where δt is infinitely small.

Exercise #8 Equation (13) is a vector equation. It therefore suffices to show that the direction and magnitude, of each side of the equation, are correct.

Direction Trivially the momentum $\vec{p}$ of a plane wave, and its wave vector $\vec{k}$, are parallel — since they both point in the direction of propagation of the plane wave.

magnitude $|\vec{p}| = \dfrac{h}{\lambda}$ → by the de-Broglie relation $\lambda = h/|\vec{p}|$

$$= \frac{h/2\pi}{\lambda/2\pi} = \underbrace{\frac{h}{2\pi}}_{\hbar} \times \underbrace{\frac{2\pi}{\lambda}}_{k} = \hbar k = |\hbar \vec{k}|$$

QED

Exercise #9 To obtain (16) from (15), we need to manipulate (15) into a form that looks like (16). In this context, note that the left side of (15) is easily modified to look like the right side of (16), if one multiplies by $(2\pi)^{-3/2}\exp(-i(\vec{k}\cdot\vec{r}-\omega t))$ and then integrates over $\vec{r}$.

So, begin with (15), which we write as:

$$\text{(A)}\quad \Psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\iiint \tilde{\Psi}(\vec{k}')\, e^{i(\vec{k}'\cdot\vec{r}-\omega' t)}\, d\vec{k}'.$$

Note: we have put primes on the $\vec{k}$'s on the right side, as this is a "dummy variable" which has been integrated over. Also, since ω depends on $\vec{k}'$, this is primed too.

Next, to make the left side of (A) look like the right side of (16), multiply by $(2\pi)^{-3/2}\exp(-i(\vec{k}\cdot\vec{r}-\omega t))$

and then integrate over $\vec{r}$: not a dummy variable

(B) $$\iiint \Psi(\vec{r},t)(2\pi)^{-3/2} e^{-i(\vec{k}\cdot\vec{r}-\omega t)} d\vec{r}$$

$$= \frac{1}{(2\pi)^{3/2}} \times \frac{1}{(2\pi)^{3/2}} \iiiiii \tilde{\Psi}(\vec{k}') e^{i(\vec{k}'\cdot\vec{r}-\omega' t)} \times e^{-i(\vec{k}\cdot\vec{r}-\omega t)} d\vec{k}' d\vec{r}$$

Thus:

$$\frac{1}{(2\pi)^{3/2}} \iiint \Psi(\vec{r},t) e^{-i(\vec{k}\cdot\vec{r}-\omega t)} d\vec{r}$$

$$= \frac{1}{(2\pi)^3} \iiiiii \tilde{\Psi}(\vec{k}') e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} e^{i(\omega-\omega')t} d\vec{k}' d\vec{r}$$

$$= \frac{1}{(2\pi)^3} \iiint \Big[\tilde{\Psi}(\vec{k}') e^{i(\omega-\omega')t} \underbrace{\iiint e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} d\vec{r}} \Big] d\vec{k}'$$

If we swap the variables $\vec{k}$ and $\vec{r}$ in (17), we have $\delta(\vec{k}) = \frac{1}{(2\pi)^3}\iiint e^{i\vec{k}\cdot\vec{r}} d\vec{r}$. Now let $\vec{k}$ be replaced by $\vec{k}'-\vec{k}$, to get $(2\pi)^3 \delta(\vec{k}'-\vec{k}) = \iiint e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} d\vec{r}$

$$= \frac{1}{(2\pi)^3} \iiint \Big[\tilde{\Psi}(\vec{k}') e^{i(\omega-\omega')t} (2\pi)^3 \delta(\vec{k}'-\vec{k}) \Big] d\vec{k}'$$ ... now cancel $(2\pi)^3$

$$= \iiint \tilde{\Psi}(\vec{k}') e^{i(\omega-\omega')t} \delta(\vec{k}'-\vec{k})\, d\vec{k}$$ ... now use (19)

$$= \tilde{\Psi}(\vec{k}) e^{i(\omega-\omega)t}$$ since $\vec{k}$ and $\vec{k}'$ are forced to be equal [by the sifting property of the Dirac delta], $\omega=\omega'$.

$$= \tilde{\Psi}(\vec{k}).$$

Thus we have obtained (18). QED!

Exercise #10 In (10), we wrote a single plane wave as:

(A) $\exp[i(\vec{k}\cdot\vec{r}-\omega t)]$.

The wavefunction $\Psi(\vec{r},t)$, corresponding to the superposition of two plane waves with the same angular frequency ω (but which point in different directions) is then:

(B) $\Psi(\vec{r},t)=\exp[i(\vec{k}\cdot\vec{r}-\omega t)]+\exp[i(\vec{k}'\cdot\vec{r}-\omega t)]$.

same ω

different wave vectors

The modulus squared, of the above expression, gives the desired probability density:

(C)
$$
\begin{aligned}
\rho(\vec{r},t) &= |\Psi(\vec{r},t)|^2\\
&= |\exp(i(\vec{k}\cdot\vec{r}-\omega t))+\exp(i(\vec{k}'\cdot\vec{r}-\omega t))|^2\\
&= |\exp(i(\vec{k}\cdot\vec{r}))+\exp(i(\vec{k}'\cdot\vec{r}))|^2\\
&= 2+2\,\mathrm{Re}\left\{e^{i(\vec{k}\cdot\vec{r}-\vec{k}'\cdot\vec{r})}\right\}\\
&= 2+2\,\mathrm{Re}\left\{e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\right\}\\
&= 2\left\{1+\cos[(\vec{k}-\vec{k}')\cdot\vec{r}]\right\}.
\end{aligned}
$$

This corresponds to a time-independent series of three dimensional interference fringes.

I will leave it to you to deduce the spacing of those three-dimensional interference fringes! ☺

Exercise #11

$$i\hbar\frac{\partial}{\partial t}\underbrace{\Psi(\vec{r},t)}_{\text{plane wave}} \overset{(27)}{=} i\hbar\frac{\partial}{\partial t}\exp\left[i\left(\frac{\vec{p}\cdot\vec{r}}{\hbar}-\frac{Et}{\hbar}\right)\right]$$

$$= \underbrace{i\hbar\left\{\frac{-iE}{\hbar}\right\}}\underbrace{\exp\left[i\left(\frac{\vec{p}\cdot\vec{r}}{\hbar}-\frac{Et}{\hbar}\right)\right]}$$

$$= E\;\underbrace{\Psi(\vec{r},t)}_{\text{plane wave}} \leftarrow (27)$$

Q.E.D.

$$-i\hbar\nabla\Psi(\vec{r},t) \overset{(27)}{=} -i\hbar\nabla\exp\left[i\left(\frac{\vec{p}\cdot\vec{r}}{\hbar}-\frac{Et}{\hbar}\right)\right]$$

$$= \underbrace{-i\hbar\left\{\frac{i\vec{p}}{\hbar}\right\}}\underbrace{\exp\left[i\left(\frac{\vec{p}\cdot\vec{r}}{\hbar}-\frac{Et}{\hbar}\right)\right]}$$

$$= \vec{p}\;\Psi(\vec{r},t) \leftarrow (27)$$

Q.E.D.

Exercise #12 we want to show that $\iiint|\Psi(\vec{r},t)|^2 d\vec{r}$ is independent of time. Equivalently, we want to show that $\frac{\partial}{\partial t}\left(\iiint|\Psi(\vec{r},t)|^2 d\vec{r}\right)$ is equal to zero. Thus:

$$\frac{\partial}{\partial t}\left[\iiint|\Psi(\vec{r},t)|^2 d\vec{r}\right]$$

$$= \frac{\partial}{\partial t}\left[\iiint\Psi(\vec{r},t)\,\Psi^*(\vec{r},t)\,d\vec{r}\right]$$

I will now stop writing this explicit functional dependence

$$= \iiint\frac{\partial}{\partial t}\left[\Psi\Psi^*\right]d\vec{r}$$ ... have brought $\frac{\partial}{\partial t}$ inside integral

$$= \iiint\left[\frac{\partial\Psi}{\partial t}\Psi^* + \Psi\frac{\partial\Psi^*}{\partial t}\right]d\vec{r}$$ ... we have used the product rule

from (35), we have (A) $$\frac{\partial\Psi}{\partial t} = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2+V\right]\Psi$$

note that $V=V^*$, i.e. V is real.

... and from the complex conjugate of (A), we have:

(B) $$\frac{\partial\Psi^*}{\partial t} = \frac{1}{-i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2+V\right]\Psi^*.$$

$$= \iiint\left\{\overbrace{\left(\frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2+V\right]\Psi\right)}^{\text{from (A)}}\Psi^* + \Psi\underbrace{\left(\frac{1}{-i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2+V\right]\Psi^*\right)}_{\text{from (B)}}\right\}d\vec{r}$$ ... note that the V's cancel

$$= \frac{i\hbar}{2m}\iiint\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right)d\vec{r}$$

Hence: (C) $$\frac{\partial}{\partial t}\left[\iiint|\Psi|^2 d\vec{r}\right] = \frac{i\hbar}{2m}\iiint\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right)d\vec{r}$$

Now, $\nabla\cdot(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)$

$$= \underbrace{\nabla\cdot(\Psi^*\nabla\Psi)}_{\text{use (53), with } A=\Psi^* \text{ and } B=\Psi} - \underbrace{\nabla\cdot(\Psi\nabla\Psi^*)}_{\text{use (53), with } A=\Psi \text{ and } B=\Psi^*}$$

$$= \Psi^*\nabla^2\Psi + \cancel{\nabla\Psi^*\cdot\nabla\Psi} - \{\Psi\nabla^2\Psi^* + \cancel{\nabla\Psi\cdot\nabla\Psi^*}\}$$

(D) $$= \Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*$$

Make use of (D) to transform the right side of (C), hence:

(E) $$\frac{\partial}{\partial t}\left[\iiint|\Psi|^2 d\vec{r}\right] = \frac{i\hbar}{2m}\iiint\nabla\cdot(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)\,d\vec{r}$$

... now use (54)

$$= -\iiint\nabla\cdot\vec{j}(\vec{r},t)\,d\vec{r}$$

... now use Gauss divergence theorem, and use same notation as eqn (57) of the notes

$$= -\iint_{\partial\Omega}\vec{j}(\vec{r},t)\cdot\hat{n}\,d\sigma,$$

(where Ω is an infinitely big volume, with surface $\partial\Omega$).

Make the volume Ω so large that $\vec{j}$ vanishes everywhere on its surface. Then $-\iint_{\partial\Omega}\vec{j}\cdot\hat{n}\,d\sigma = 0$, and so:

(F) $$\frac{\partial}{\partial t}\left[\iiint|\Psi|^2 d\vec{r}\right] = 0. \quad \text{QED.}$$

(PS — This question can also be done on inspection, by examining eqn. (57) of the notes, and replacing Ω with the infinitely-large volume introduced above.)

Exercise #13 The free-particle Schrödinger equation, (36), can be written as:

(A) $$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\right)\Psi = 0.$$ We want to show that Ψ, as given in (15), obeys this equation. Thus:

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\right)\Psi$$ ... now use (15)

$$= \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\right)\left\{\frac{1}{(2\pi)^{3/2}}\iiint\tilde{\Psi}(\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)}d\vec{k}\right\}$$

(... now bring the operator $(i\hbar\,\partial/\partial t + \frac{\hbar^2}{2m}\nabla^2)$ inside the integral, noting that it acts only on $\exp(i(\vec{k}\cdot\vec{r} - \omega t))$)

$$= \frac{1}{(2\pi)^{3/2}} \iiint \tilde{\Psi}(\vec{k}) \left\{ \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 \right) \exp[i(\vec{k}\cdot\vec{r} - \omega t)] \right\} d\vec{k}$$

$$= \frac{1}{(2\pi)^{3/2}} \iiint \tilde{\Psi}(\vec{k}) \left\{ \underbrace{i\hbar(-\omega i) + \frac{\hbar^2}{2m}(i\vec{k})\cdot(i\vec{k})} \right\} \exp[i(\vec{k}\cdot\vec{r} - \omega t)]\, d\vec{k}$$

this equals $\hbar\omega - \frac{|\hbar\vec{k}|^2}{2m}$

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... which equals $E - \frac{|\vec{p}|^2}{2m}$

See footnote #9

which equals zero (since $E = \frac{|\vec{p}|^2}{2m}$ for a free particle)

$$= \frac{1}{(2\pi)^{3/2}} \iiint \tilde{\Psi}(\vec{k}) \times 0 \times \exp[i(\vec{k}\cdot\vec{r} - \omega t)]\, d\vec{k} = 0 \quad \text{Q.E.D.}$$

Exercise #14 We show that 52 leads to 51:

From 52, $\frac{\partial}{\partial t}|\Psi|^2 + \nabla\cdot\left(\frac{i\hbar}{2m}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)\right) = 0$

$$\Rightarrow \frac{\partial}{\partial t}(\Psi^*\Psi) + \frac{i\hbar}{2m}\nabla\cdot(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}(\Psi^*\Psi)}_{\text{use product rule}} + \frac{i\hbar}{2m}\left\{ \underbrace{\nabla\cdot(\Psi\nabla\Psi^*)}_{\text{use (53) with } A=\Psi \text{ and } B=\Psi^*} - \underbrace{\nabla\cdot(\Psi^*\nabla\Psi)}_{\text{use (53), with } A=\Psi^* \text{ and } B=\Psi} \right\} = 0$$

$$\Rightarrow \underbrace{\frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t}} + \frac{i\hbar}{2m}\left\{ \Psi\nabla^2\Psi^* + \cancel{\nabla\Psi\cdot\nabla\Psi^*} - \left(\Psi^*\nabla^2\Psi + \cancel{\nabla\Psi^*\cdot\nabla\Psi}\right) \right\} = 0$$

take these terms over to the right side. Multiply the resulting equation by $-i\hbar$, and you will obtain 51. Q.E.D.

PS – the logic, for getting from 51 to 52, is similar!

Exercise #15 Substitute (58) into (54). Hence:

$$\vec{j} = \frac{i\hbar}{2m}\left\{\Psi \nabla \Psi^* - \Psi^* \nabla \Psi\right\}$$

$$= \frac{i\hbar}{2m}\left\{\sqrt{\rho}\, e^{iS} \nabla \left[\sqrt{\rho}\, e^{iS}\right]^* - \left[\sqrt{\rho}\, e^{iS}\right]^* \nabla \left[\sqrt{\rho}\, e^{iS}\right]\right\}$$

$$= \frac{i\hbar}{2m}\left\{\sqrt{\rho}\, e^{iS} \nabla \left[\sqrt{\rho}\, e^{-iS}\right] - \sqrt{\rho}\, e^{-iS} \nabla \left[\sqrt{\rho}\, e^{iS}\right]\right\}$$

(... now use "product rule" for ∇, namely
$\nabla(AB) = A\nabla B + B\nabla A$)

$$= \frac{i\hbar}{2m}\left\{\sqrt{\rho}\, e^{iS}\left[\sqrt{\rho}\, \nabla e^{-iS} + e^{-iS} \nabla\sqrt{\rho}\right] - \sqrt{\rho}\, e^{-iS}\left[\sqrt{\rho}\, \nabla e^{iS} + e^{iS} \nabla\sqrt{\rho}\right]\right\}$$

$$\left(\nabla e^{-iS} = -i(\nabla S)e^{-iS}\right)$$

("chain rule" for ∇ implies $\nabla\sqrt{\rho} = \nabla(\rho^{\frac{1}{2}}) = \frac{1}{2}\rho^{-\frac{1}{2}}\nabla\rho$)

$$\left(\nabla e^{iS} = i(\nabla S)e^{iS}\right)$$

$$= \frac{i\hbar}{2m}\left\{\sqrt{\rho}\, e^{iS}\left[\sqrt{\rho}(-i)(\nabla S)e^{-iS} + e^{-iS}\left(\tfrac{1}{2}\right)\left(\rho^{-\frac{1}{2}}\right)\nabla\rho\right] - \sqrt{\rho}\, e^{-iS}\left[\sqrt{\rho}\, i(\nabla S)e^{iS} + e^{iS}\left(\tfrac{1}{2}\rho^{-\frac{1}{2}}\nabla\rho\right)\right]\right\}$$

... some terms now cancel ...

$$= \frac{i\hbar}{2m}\sqrt{\rho}\left\{-i\sqrt{\rho}\,\nabla S + \cancel{\tfrac{1}{2}\rho^{-\frac{1}{2}}\nabla\rho} - i\sqrt{\rho}\,\nabla S - \cancel{\tfrac{1}{2}\rho^{-\frac{1}{2}}\nabla\rho}\right\}$$

$$= \frac{i\hbar}{2m}\sqrt{\rho} \times \left(-2i\sqrt{\rho}\,\nabla S\right) = \frac{\hbar\rho\nabla S}{m}.$$ ← This is the required result.

Thus $$\boxed{\vec{j}(\vec{r},t) = \frac{\hbar}{m}\rho(\vec{r},t)\,\underline{\nabla S(\vec{r},t)}}$$

indicates that the current vector is perpendicular to the surfaces of the constant phase.

current vector is proportional to probability density at $(\vec{r},t)$ — this is intuitively reasonable!

Exercise #16 Substitute (44) into (54). Thus, for monoenergetic wave function of the form (44), the current density vector $\vec{j}(\vec{r},t)$ is:

$$\vec{j}(\vec{r},t) \overset{(54)}{=} \frac{i\hbar}{2m}\left[\Psi_E(\vec{r},t)\nabla\Psi^*(\vec{r},t) - \Psi^*(\vec{r},t)\nabla\Psi(\vec{r},t)\right]$$

$$\overset{(44)}{=} \frac{i\hbar}{2m}\left[\psi(\vec{r})\cancel{e^{-iEt/\hbar}}\nabla\left\{\psi^*(\vec{r})\cancel{e^{+iEt/\hbar}}\right\} - \psi^*(\vec{r})\cancel{e^{+iEt/\hbar}}\nabla\left\{\psi(\vec{r})\cancel{e^{-iEt/\hbar}}\right\}\right]$$

$$= \frac{i\hbar}{2m}\left[\psi(\vec{r})\nabla\psi^*(\vec{r}) - \psi^*(\vec{r})\nabla\psi(\vec{r})\right];$$

this current vector is independent of time, as required.

Exercise #17 Begin with expression (54) for the probability current:

$$(54)\quad \vec{j} = \frac{i\hbar}{2m}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right).$$

Now, if Ψ is real, then $\Psi = \Psi^*$, and so the above equation becomes:

$$\vec{j} = \frac{i\hbar}{2m}\left(\Psi\nabla\Psi - \Psi\nabla\Psi\right) = \vec{0}.$$

Thus the current density vanishes everywhere, for any wave function that is a real function of position and time.

Exercise #18 (a) Momentum operator. From (68), we need to show that $\iiint \Psi^*\vec{p}\,\Psi\, d\vec{r} = \iiint (\vec{p}\Psi)^*\Psi\, d\vec{r}$. — (A)

Here, we just consider the x component of (A), as the proof for the y and z components is analogous. Thus we need to show:

(B) $\iiint \Psi^* p_x \Psi\, d\vec{r} = \iiint (p_x\Psi)^*\Psi\, d\vec{r}$. Now,

$$\iiint \Psi^* p_x\Psi\, d\vec{r} = \iiint_{-\infty}^{\infty} \Psi^*(-i\hbar\partial_x)\Psi\, dx\,dy\,dz \quad \text{(see (31))}$$

$$= -i\hbar\iint_{-\infty}^{\infty} dy\,dz\left\{\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx\right\}$$

now use integration by parts

$$= -i\hbar\iint_{-\infty}^{\infty} dy\,dz\left\{\underbrace{\Psi^*\Psi\Big|_{-\infty}^{\infty}}_{\text{vanishes, since } |\Psi|\to 0 \text{ at infinity, by assumption}} - \int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\Psi\, dx\right\}$$

$$= i\hbar\iiint\frac{\partial\Psi^*}{\partial x}\Psi\, dx\,dy\,dz$$

$$= \iiint(-i\hbar\partial_x\Psi)^*\Psi\, d\vec{r} = \iiint(p_x\Psi)^*\Psi\, d\vec{r} \quad \text{Q.E.D.}$$

(b) Energy operator. In exercise #12, we showed that the Schrödinger equation implies eqn (22) to hold, for the case of a normalised wave function:

Ⓐ $\iiint |\Psi|^2 d\vec{r} = 1$ (for any time t). Thus:

Ⓑ $\iiint \Psi^* \Psi d\vec{r} = 1$. Now apply ∂_t to both sides:

Ⓒ $\partial_t \iiint \Psi^* \Psi d\vec{r} = 0$. Bring ∂_t inside integral & use the product rule, so that:

Ⓓ $\iiint \left(\Psi^* \frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi^*}{\partial t}\right) d\vec{r} = 0$. Therefore:

$$\iiint \Psi^* \partial_t \Psi \, d\vec{r} = -\iiint \Psi (\partial_t \Psi)^* d\vec{r}$$

$$\iiint \Psi^* (\underbrace{i\hbar}\partial_t) \Psi \, d\vec{r} = -\iiint \Psi (i\hbar)(\partial_t \Psi)^* d\vec{r}$$

$$\iiint \Psi^* \underbrace{(i\hbar \partial_t)}_{E} \Psi \, d\vec{r} = \iiint \Psi \underbrace{(+i\hbar \partial_t \Psi)^*}_{E} d\vec{r}$$

$$\iiint \Psi^* E \Psi \, d\vec{r} = \iiint (E\Psi)^* \Psi \, d\vec{r}$$

From (68), we conclude that E is hermitian. QED.

<u>Exercise #19</u> ⓐ $[A,B] \equiv AB - BA = -(BA - AB) \equiv -[B,A]$ □

ⓑ $[A,B+C] \equiv A(B+C) - (B+C)A = AB + AC - BA - CA$

$$= \underbrace{AB - BA} + \underbrace{AC - CA}$$

$$\equiv [A,B] + [A,C] \quad \square$$

ⓒ $[A,B]C + B[A,C] \equiv (AB-BA)C + B(AC-CA)$

$$= ABC - \cancel{BAC} + \cancel{BAC} - BCA$$

$$= ABC - BCA$$

$$= A(BC) - (BC)A \equiv [A,BC] \quad \square$$

ⓓ $[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$

$$\equiv [A, BC - CB] + [B, CA - AC] + [C, AB - BA]$$

$$\equiv A(BC-CB)-(BC-CB)A$$
$$+B(CA-AC)-(CA-AC)B$$
$$+C(AB-BA)-(AB-BA)C$$

... now expand it all out!

$$=\cancel{ABC}-ACB-\cancel{BCA}+CBA$$
$$+\cancel{BCA}-BAC-\cancel{CAB}+\cancel{ACB}$$
$$+\cancel{CAB}-CBA-\cancel{ABC}+BAC=0$$

Note – in all of the above, the order of multiplication (i.e., the order in which factors are multiplied.) has always been maintained.

<u>Exercise #20</u> (a) $[x, p_x]\Psi$ note!!

$$=[x, -i\hbar\partial_x]\Psi = \{x(-i\hbar\partial_x)-(-i\hbar\partial_x)x\}\Psi$$

$$=-i\hbar\{x\partial_x-\partial_x x\}\Psi$$

$$=-i\hbar\{x\partial_x\Psi-\partial_x x\Psi\}$$

$$=-i\hbar\left\{x\frac{\partial\Psi}{\partial x}-\frac{\partial}{\partial x}(x\Psi)\right\}$$

now use product rule:

$$\left(\frac{\partial}{\partial x}(x\Psi)=x\frac{\partial\Psi}{\partial x}+\frac{\partial x}{\partial x}\Psi = x\frac{\partial\Psi}{\partial x}+\Psi\right)$$

$$=-i\hbar\left\{\cancel{x\frac{\partial\Psi}{\partial x}}-\cancel{x\frac{\partial\Psi}{\partial x}}-\Psi\right\}$$

$=i\hbar\Psi$. Since $[x, p_x]\Psi = i\hbar\Psi$ for arbitrary Ψ, we conclude that $[x, p_x] = i\hbar$. QED.

Proofs, for the remaining bits of equations (71a) and (71b), are analogous, and will not be written here! ☺

<u>Exercise #21</u>

Let us consider two observables A and B. Let $\langle A\rangle \equiv \langle\Psi|A|\Psi\rangle$ be the expectation value of A in a given state Ψ(normalised to unity) and let $\langle B\rangle \equiv \langle\Psi|B|\Psi\rangle$ be the expectation value of B in the state Ψ. We define the uncertainty ΔA to be

$$\Delta A = [\langle(A-\langle A\rangle)^2\rangle]^{1/2} \qquad [5.99]$$

so that

$$(\Delta A)^2 = \langle(A-\langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2 \qquad [5.100]$$

is the mean-square deviation about the expectation value $\langle A\rangle$. Similarly, we define the uncertainty ΔB to be

$$\Delta B=[\langle(B-\langle B\rangle)^2\rangle]^{1/2}. \qquad [5.101]$$

We shall now prove that

$$\boxed{\Delta A\Delta B\geqslant\tfrac{1}{2}|\langle[A,B]\rangle|.} \qquad [5.102]$$

To this end, we first introduce the linear Hermitian operators

$$\bar{A}=A-\langle A\rangle, \qquad \bar{B}=B-\langle B\rangle \qquad [5.103]$$

which are such that their expectation values vanish. In terms of these operators, we have

$$(\Delta A)^2=\langle\bar{A}^2\rangle, \qquad (\Delta B)^2=\langle\bar{B}^2\rangle \qquad [5.104]$$

and we also note that

$$[\bar{A},\bar{B}]=[A-\langle A\rangle,B-\langle B\rangle]=[A,B]. \qquad [5.105]$$

Next, we consider the linear (but not Hermitian) operator

$$C=\bar{A}+\mathrm{i}\lambda\bar{B} \qquad [5.106]$$

where λ is a real constant. The adjoint of C is the operator $C^\dagger=\bar{A}-\mathrm{i}\lambda\bar{B}$ and we note that the expectation value of $CC^\dagger$ is real and non-negative, since

$$\langle CC^\dagger\rangle=\langle\Psi|CC^\dagger|\Psi\rangle=\langle C^\dagger\Psi|C^\dagger\Psi\rangle\geqslant 0. \qquad [5.107]$$

From [5.106] and [5.107] it follows that the expectation value

$$\langle(\bar{A}+\mathrm{i}\lambda\bar{B})(\bar{A}-\mathrm{i}\lambda\bar{B})\rangle=\langle\bar{A}^2+\lambda^2\bar{B}^2-\mathrm{i}\lambda[\bar{A},\bar{B}]\rangle \qquad [5.108]$$

is real and non-negative. Using [5.108], [5.104] and [5.105], we see that the function

$$\begin{aligned}f(\lambda)&=\langle\bar{A}^2\rangle+\lambda^2\langle\bar{B}^2\rangle-\mathrm{i}\lambda\langle[\bar{A},\bar{B}]\rangle\\&=(\Delta A)^2+\lambda^2(\Delta B)^2-\mathrm{i}\lambda\langle[A,B]\rangle\end{aligned} \qquad [5.109]$$

is also real and non-negative, which implies that $\langle[A,B]\rangle$ is purely imaginary. Now, the function $f(\lambda)$ has a minimum for

$$\lambda_0=\frac{\mathrm{i}}{2}\frac{\langle[A,B]\rangle}{(\Delta B)^2} \qquad [5.110]$$

and the value of $f(\lambda)$ at the minimum is

$$f(\lambda_0)=(\Delta A)^2+\frac{1}{4}\frac{(\langle[A,B]\rangle)^2}{(\Delta B)^2}. \qquad [5.111]$$

Since this value is non-negative, we must have

$$(\Delta A)^2(\Delta B)^2\geqslant-\frac{1}{4}(\langle[A,B]\rangle)^2 \qquad [5.112]$$

and the property [5.102] follows by remembering that $\langle[A,B]\rangle$ is purely imaginary.

For two observables which are canonically conjugate, so that $[A,B]=\mathrm{i}\hbar$, we have $\langle[A,B]\rangle=\mathrm{i}\hbar$ and we therefore deduce from [5.112] that

$$\Delta A\Delta B\geqslant\frac{\hbar}{2}. \qquad [5.113]$$

In particular, for the pairs of canonically conjugate variables (x,p_x), (y,p_y) and (z,p_z), we can state the position–momentum uncertainty relations in the precise form

$$\Delta x\Delta p_x\geqslant\frac{\hbar}{2}, \qquad \Delta y\Delta p_y\geqslant\frac{\hbar}{2}, \qquad \Delta z\Delta p_z\geqslant\frac{\hbar}{2} \qquad [5.114]$$

with

$$\Delta x=[\langle(x-\langle x\rangle)^2\rangle]^{1/2}, \qquad \Delta p_x=[\langle(p_x-\langle p_x\rangle)^2\rangle]^{1/2} \qquad [5.115]$$

and similar definitions for Δy, Δp_y, Δz and Δp_z.

Exercise #22 From (35), we have:

$$\text{(A)}\quad \frac{\partial \Psi}{\partial t} = \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\Psi$$, the complex conjugate of which is:

$$\text{(B)}\quad \frac{\partial \Psi^*}{\partial t} = \frac{-1}{i\hbar}\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\Psi^*.$$

Note V assumed real, so $V = V^*$.

Hence (76) becomes:

$$(76) \Rightarrow \frac{d}{dt}\langle p_x\rangle = -i\hbar\iiint\left(\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\frac{\partial \Psi}{\partial t}\right)d\vec{r}$$

$$= -i\hbar\iiint\left\{\left(\frac{-1}{i\hbar}\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\Psi^*\right)\frac{\partial \Psi}{\partial x}\right\}d\vec{r}$$

$$-i\hbar\iiint \Psi^*\frac{\partial}{\partial x}\left(\left(\frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\right)\Psi\right)d\vec{r}$$

$$= \iiint\left\{\frac{\partial \Psi}{\partial x}\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\Psi^* - \Psi^*\frac{\partial}{\partial x}\left(\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\Psi\right)\right\}d\vec{r}$$

now expand out

$$= \iiint\left\{\frac{-\hbar^2}{2m}\frac{\partial \Psi}{\partial x}\left(\nabla^2\Psi^*\right) + \left(\frac{\partial \Psi}{\partial x}\right)V\Psi^* + \frac{\hbar^2}{2m}\Psi^*\frac{\partial}{\partial x}\left(\nabla^2\Psi\right) - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right\}d\vec{r}$$

$$= \iiint\left\{\frac{\partial \Psi}{\partial x}V\Psi^* - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right\}d\vec{r}$$

$$+\frac{\hbar^2}{2m}\iiint\left\{\Psi^*\frac{\partial}{\partial x}\left(\nabla^2\Psi\right) - \frac{\partial \Psi}{\partial x}\left(\nabla^2\Psi^*\right)\right\}d\vec{r}$$

We claim that the second integral vanishes, in the above expression. (Denote this integral by "I", for later reference). Thus:

$$\frac{d}{dt}\langle p_x\rangle = \iiint\left\{\frac{\partial \Psi}{\partial x}V\Psi^* - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right\}d\vec{r}$$

use product rule

$$= \iiint\left\{\cancel{\frac{\partial \Psi}{\partial x}V\Psi^*} - \Psi^*\frac{\partial V}{\partial x}\Psi - \cancel{\Psi^* V\frac{\partial \Psi}{\partial x}}\right\}d\vec{r}$$

$$\text{(C)}\quad = +\iiint \Psi^*\left(\frac{-\partial V}{\partial x}\right)\Psi\, d\vec{r} \equiv -\left\langle\frac{\partial V}{\partial x}\right\rangle$$, as required.

To complete the proof, we need to show that $I=0$:

$$I \equiv \iiint \left\{ \Psi^* \frac{\partial}{\partial x}(\nabla^2 \Psi) - \frac{\partial \Psi}{\partial x}(\nabla^2 \Psi^*) \right\} d\vec{r}$$

$$= \iiint \left\{ \Psi^* \nabla^2 \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \nabla^2 \Psi^* \right\} d\vec{r} \qquad \text{(D)}$$

Now, $\nabla \cdot \left(\Psi^* \nabla \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \nabla \Psi^* \right)$

$$= \nabla \cdot \left(\Psi^* \nabla \frac{\partial \Psi}{\partial x} \right) - \nabla \cdot \left(\frac{\partial \Psi}{\partial x} \nabla \Psi^* \right)$$

(now use (53) ... also, cf top of page 15 of these solutions)

$$= \cancel{\nabla \Psi^* \cdot \nabla \frac{\partial \Psi}{\partial x}} + \Psi^* \nabla^2 \frac{\partial \Psi}{\partial x} - \cancel{\left(\nabla \frac{\partial \Psi}{\partial x}\right) \cdot \nabla \Psi^*} - \frac{\partial \Psi}{\partial x} \nabla^2 \Psi^*$$

$$= \Psi^* \nabla^2 \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \nabla^2 \Psi^*,$$

which is the same as the term in braces in the lowest line of (D). Hence (D) becomes:

$$I = \iiint \nabla \cdot \left(\Psi^* \nabla \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \nabla \Psi^* \right) d\vec{r}$$

(... now use Gauss divergence theorem, with volume ω that is taken to fill the whole space. Use notation from page 15 of lecture notes.)

$$= \oiint_{\partial \omega} \underbrace{\left(\Psi^* \nabla \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \nabla \Psi^* \right)}_{\text{wavefunction assumed to vanish at infinity, hence this is zero}} \cdot \hat{n} \, d\sigma$$

$$= 0 \qquad \text{QED.}$$

<u>Exercise #23</u>

$$m \frac{d}{dt} \langle \vec{r} \rangle \equiv m \frac{d}{dt} \iiint \Psi^* \vec{r} \Psi \, d\vec{r}$$

$$= m \iiint \frac{d}{dt} (\Psi^* \vec{r} \Psi) \, d\vec{r}$$

now use product rule, i.e.

$$\frac{d}{dt}(abc) = \frac{da}{dt} bc + a \frac{db}{dt} c + ab \frac{dc}{dt}$$

$$m\frac{d}{dt}\langle\vec{r}\rangle$$

$$= m\iiint\left\{\underbrace{\frac{\partial\Psi^*}{\partial t}}_{\text{use (B) on page 22 of these solutions}}\vec{r}\,\Psi + \Psi^*\frac{d\vec{r}}{dt}\Psi + \Psi^*\vec{r}\underbrace{\frac{\partial\Psi}{\partial t}}_{\text{use (A) on p 22 of these solutions}}\right\}d\vec{r}$$

$$= \iiint\Psi^*\left(m\frac{d\vec{r}}{dt}\right)\Psi\,d\vec{r}$$

$$+ m\iiint\left(\frac{-1}{i\hbar}\right)\left(\frac{-\hbar^2}{2m}\nabla^2\Psi^* + V\Psi^*\right)\vec{r}\,\Psi\,d\vec{r}$$

$$+ m\iiint\Psi^*\vec{r}\left(\frac{1}{i\hbar}\right)\left(\frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi\right)d\vec{r}$$

note that V cancels, here

$$= \iiint\Psi^*\left(m\underbrace{\frac{d\vec{r}}{dt}}\right)\Psi\,d\vec{r} + \frac{\hbar}{2i}\iiint\left\{(\nabla^2\Psi^*)\vec{r}\,\Psi - \Psi^*\vec{r}\,\nabla^2\Psi\right\}d\vec{r}$$

this vanishes, because the position vector $\vec{r}$ has no t dependence

$$= \frac{\hbar}{2i}\iiint\left\{(\nabla^2\Psi^*)\vec{r}\,\Psi - \Psi^*\vec{r}\,\nabla^2\Psi\right\}d\vec{r} \qquad (1)$$

We now restrict consideration to the <u>x component</u> of equation (1), noting that the treatment of the y and z components is quite analogous.

Now, the x component of (1) — obtained by replacing $\vec{r}$ by x, is:

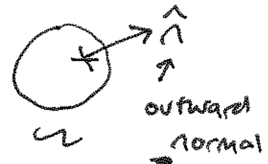
$$(2)\quad m\frac{d}{dt}\langle x\rangle = \frac{\hbar}{2i}\iiint\left\{(\nabla^2\Psi^*)x\,\Psi - \Psi^* x\,\nabla^2\Psi\right\}d\vec{r}.$$

Put this to one side for the moment. Recall Green's theorem from your studies on vector analysis:

$$(3)\quad \underbrace{\iiint_{\Omega}}_{\text{volume}}(f\nabla^2 g - g\nabla^2 f)\,d\vec{r} = \underbrace{\oiint_{\partial\Omega}}_{\text{boundary of volume }\Omega}(f\,\partial_n g - g\,\partial_n f)\,d\sigma$$

∂_n → derivative with respect to $\hat{n}$

$d\sigma$ → areal element



(Please see a vector analysis text if (3) is unclear to you! ☺) In (3), make the identifications:

$$(4)\quad f = \Psi^* \quad\text{and}\quad g = x\Psi,$$

so that Green's theorem (3) becomes:

(5) $$\iiint_{\Omega} \left(\Psi^* \nabla^2 (x\Psi) - x\Psi \nabla^2 \Psi^*\right) d\vec{V} = \oiint_{\partial\Omega} \left(\Psi^* \partial_n (x\Psi) - x\Psi \partial_n \Psi^*\right) d\sigma.$$

Now perform the usual "trick", of making the volume of integration Ω fill the whole of 3D space. Then the surface integral vanishes, on the right side of (5) (since the wave function vanishes at infinity, by assumption).

Hence (5) becomes:

(6) $$\iiint \Psi^* \nabla^2 (x\Psi)\, d\vec{r} = \iiint x\Psi \nabla^2 \Psi^*\, d\vec{r}.$$

Same as the first integral in (2)!

The right side of (6) is identical to the first integral in equation (2). Hence we may rewrite (2) as:

(7) $$m\frac{d}{dt}\langle x\rangle = \frac{\hbar}{2i}\iiint \left\{\Psi^* \nabla^2 (x\Psi) - \Psi^* x \nabla^2 \Psi\right\} d\vec{r}$$

$$= \frac{\hbar}{2i}\iiint \Psi^* \left\{\nabla^2 (x\Psi) - x\nabla^2\Psi\right\} d\vec{r}$$

$$\left(\nabla^2 (x\Psi) = x\nabla^2\Psi + \Psi\nabla^2 x + 2\nabla x\cdot\nabla\Psi\right)$$

$$= \frac{\hbar}{2i}\iiint \Psi^* \left\{\cancel{x\nabla^2\Psi} + \Psi\underbrace{\nabla^2 x}_{\text{zero}} + 2\nabla x\cdot\nabla\Psi - \cancel{x\nabla^2\Psi}\right\} d\vec{r}$$

$$= \frac{\hbar}{i}\iiint \Psi^* (\nabla x\cdot\nabla\Psi)\, d\vec{r}$$

Now, $\nabla x = \hat{x}$, where "$\hat{x}$" denotes a unit vector that points in the positive x direction. Hence (7) becomes:

(8) $$m\frac{d}{dt}\langle x\rangle = \frac{\hbar}{i}\iiint \Psi^* (\underbrace{\hat{x}\cdot\nabla}_{\partial/\partial x})\Psi\, d\vec{r}$$

$$= \frac{\hbar}{i}\iiint \Psi^* \frac{\partial\Psi}{\partial x}\, d\vec{r}$$

$$= \iiint \Psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\, d\vec{r}$$

$P_x \equiv x$ component of momentum operator (see eqn. (31) of notes)

$$= \iiint \Psi^* P_x \Psi\, d\vec{r}$$

$$= \langle P_x\rangle \quad \text{(cf. (65) of notes)}$$

Exercise #24 Recall that $A^\dagger = A$ for a Hermitian operator, and that $(DB)^\dagger = B^\dagger D^\dagger$.
Also, recall that both x and p_x are Hermitian. see (141) of the notes!

(a) We need to show that $(x^m p_x^n)^\dagger \neq x^m p_x^n$.

Now,
$$(x^m p_x^n)^\dagger = (p_x^n)^\dagger (x^m)^\dagger$$
$$= p_x^n x^m \quad \text{since } p_x \text{ and } x \text{ are Hermitian}$$
$$\neq x^m p_x^n \quad \text{since } x \text{ and } p_x \text{ do not commute.}$$

(b) We need to show that:
$$\left[\tfrac{1}{2}(x^m p_x^n + p_x^n x^m)\right]^\dagger = \tfrac{1}{2}(x^m p_x^n + p_x^n x^m).$$

Now,
$$\left[\tfrac{1}{2}(x^m p_x^n + p_x^n x^m)\right]^\dagger$$
$$= \tfrac{1}{2}(x^m p_x^n)^\dagger + \tfrac{1}{2}(p_x^n x^m)^\dagger$$
$$= \tfrac{1}{2}(p_x^n)^\dagger (x^m)^\dagger + \tfrac{1}{2}(x^m)^\dagger (p_x^n)^\dagger$$
$$= \tfrac{1}{2}(p_x^\dagger)^n (x^\dagger)^m + \tfrac{1}{2}(x^\dagger)^m (p_x^\dagger)^n$$
$$= \tfrac{1}{2}(p_x)^n (x)^m + \tfrac{1}{2}(x)^m (p_x)^n \quad \text{since } p_x \text{ and } x \text{ are Hermitian}$$
$$= \tfrac{1}{2}(x^m p_x^n + p_x^n x^m) \quad \text{as required.}$$

Exercise #25 When the potential is $V(\vec{r})$, the time-independent Schrödinger equation ("TISE") (47) reads:

(a) $$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi_1(\vec{r}) = E\,\psi_1(\vec{r})$$

When the potential is $V(\vec{r}) + V_0$, the TISE reads:

(b) $$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(\vec{r}) + V_0\right)\psi_2(\vec{r}) = E_2\,\psi_2(\vec{r})$$

Note — the wave functions in (a) and (b) have been respectively labelled $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$, to allow for the logical possibility that these wave functions are different. The energies have been called E and E_2, for the same reason.

Now, (b) can be re-written as:

(c) $$\left(-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r})\right)\psi_2(\vec{r})=(E_2-V_0)\psi_2(\vec{r})$$

The left side has the same functional form as (a). Assuming identical boundary conditions (this is physically reasonable) the energy eigenfunctions will be the same as (a); the only difference is that the "new" energy eigenvalues satisfy:

(d) $$E_2-V_0=E \Rightarrow E_2=E+V_0.$$

Thus the energy eigenvalues have been shifted from E to $E+V_0$; the corresponding spatial wave functions are unchanged.

<u>Exercise #26</u> Note: You may want to re-read §5 before studying the solution below!

Let us begin with the $t=0$ case of (16):

(a) $$\tilde{\Psi}(\vec{k})=\frac{1}{(2\pi)^{3/2}}\iiint\Psi(\vec{r}',t=0)e^{-i\vec{k}\cdot\vec{r}'}d\vec{r}'.$$

Note that we have put primes on the "dummy" variable of integration ($\vec{r}\to\vec{r}'$).

Next, throw (a) into equation (15) of the notes, to give a formula that serves to transform $\Psi(\vec{r},t=0)$ into $\Psi(\vec{r},t)$:

(b) $$\Psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\iiint\left\{\frac{1}{(2\pi)^{3/2}}\iiint\Psi(\vec{r}',t=0)e^{-i\vec{k}\cdot\vec{r}'}d\vec{r}'\right\}e^{i(\vec{k}\cdot\vec{r}-\omega t)}d\vec{k}$$

note: no prime here!

Now interchange the order of integration:

$$\text{(c)}\quad \Psi(\vec{r},t) = \frac{1}{(2\pi)^3}\iiint d\vec{r}\,'\left\{\Psi(\vec{r}\,',t=0)\iiint d\vec{k}\left[e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}e^{-i\omega t}\right]\right\}$$

Note $\omega = \omega(\vec{k})$, hence it is under the integral signs.

$$\text{(d)}\quad \Psi(\vec{r},t) = \iiint d\vec{r}\,'\left\{\Psi(\vec{r}\,',t=0)\underbrace{\left[\frac{1}{(2\pi)^3}\iiint d\vec{k}\, e^{i\{\vec{k}\cdot(\vec{r}-\vec{r}\,')-\omega t\}}\right]}_{\text{call this object } K(\vec{r}-\vec{r}\,',t)}\right\}$$

$$\equiv \iiint d\vec{r}\,'\,\Psi(\vec{r}\,',t=0)K(\vec{r}-\vec{r}\,',t)$$

This is the required result (for $t_1 = 0$ and $t_2 = t$).

Exercise #27 (a) The specialisation of equation (54) to one spatial dimension, is:

$$\text{(a)}\quad j(x,t) = \frac{i\hbar}{2m}\left[\Psi(x,t)\frac{\partial}{\partial x}\Psi^*(x,t) - \Psi^*(x,t)\frac{\partial}{\partial x}\Psi(x,t)\right]$$

(b) write a stationary state wave function, in one spatial dimension, as:

$$\text{(b)}\quad \Psi(x,t) = \psi(x)\exp(-iEt/\hbar). \qquad \text{(see (44))}$$

(... and (89)!)

Substitute (b) into (a), hence:

$$\text{(c)}\quad j = \frac{i\hbar}{2m}\left[\psi \cancel{e^{-iEt/\hbar}}\frac{\partial}{\partial x}\psi^*\cancel{e^{iEt/\hbar}} - \psi^*\cancel{e^{iEt/\hbar}}\frac{\partial}{\partial x}\psi\cancel{e^{-iEt/\hbar}}\right]$$

... time has now dropped out!

$$= \frac{i\hbar}{2m}\left[\psi\frac{\partial}{\partial x}\psi^* - \psi^*\frac{\partial}{\partial x}\psi\right]$$

This shows us that j is independent of time (an analogue of steady state flow in classical fluid mechanics) ... while this is interesting, the question asks us to show that j is independent of position!!

To this end, let us write down the one-dimensional version of the continuity equation (55):

(d) $\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} j(x,t) = 0.$

we can now drop the "t", since we saw earlier that j is independent of time.

Now, from the squared modulus of (b):

(e) $|\Psi(x,t)|^2 \overset{(20)}{=} \rho(x,t) = |\psi(x)\exp(-iEt/\hbar)|^2$
$= |\psi(x)|^2$

We see that ρ is independent of time. Hence, $\frac{\partial}{\partial t}\rho = 0$, and so (d) becomes:

(f) $\frac{\partial}{\partial x} j(x) = 0.$

Thus the current density is independent of position.

Exercise #28 Let (a) $k = \sqrt{2mE}/\hbar$, so that (86) becomes:

(b) $\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)$. One can then very easily verify that (87) solves (b).

Regarding the corresponding $j(x)$, substitute (87) from the lecture notes into eqn. (c) on p. 28 of these notes, so:

(c) $$j(x) = \frac{i\hbar}{2m}\left[\psi\frac{\partial}{\partial x}\psi^* - \psi^*\frac{\partial}{\partial x}\psi\right]$$

$$= \frac{i\hbar}{2m}\left[\left(Ae^{ikx}+Be^{-ikx}\right)\frac{\partial}{\partial x}\left(A^*e^{-ikx}+B^*e^{ikx}\right) - \left(A^*e^{-ikx}+B^*e^{ikx}\right)\frac{\partial}{\partial x}\left(Ae^{ikx}+Be^{-ikx}\right)\right]$$

$$= \frac{i\hbar}{2m}\left[\left(Ae^{ikx}+Be^{-ikx}\right)\left(A^*(-ik)e^{-ikx}+B^*(ik)e^{ikx}\right) - \left(A^*e^{-ikx}+B^*e^{ikx}\right)\left(A(ik)e^{ikx}+B(-ik)e^{-ikx}\right)\right]$$

"now expand"

$$= \frac{i\hbar}{2m}\left[-ik|A|^2 + \cancel{ikAB^*e^{2ikx}} + \cancel{A^*B(-ik)e^{-2ikx}} + ik|B|^2 - ik|A|^2 + \cancel{ikA^*Be^{-2ikx}} - \cancel{ikAB^*e^{2ikx}} + ik|B|^2\right]$$

$$= \frac{\hbar k}{m}\left[|A|^2 - |B|^2\right]$$

Limit cases:

(a) When $B=0$, the wavefunction is Ae^{ikx} (see (87)), namely a 1D plane wave that travels to the right. The current $j(x)$, on the previous page, reduces to $\hbar k|A|^2/m$. Thus, the larger the magnitude of A, the larger the current. The fact, that the current is greater than zero, indicates the wave travels to the right.

(b) When $A=0$, $\psi(x) = Be^{-ikx}$, which is a plane wave that travels to the left. The current $j(x)$ is now negative (i.e., $-\hbar k|B|^2/m$), indicating that the energy indeed flows in the negative x ("left") direction.

Exercise #29 (a) In region II, where $V(x)=V_0$ (see FIG 10), the equation (85) becomes:

$$\text{(a)}\quad \left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}+V_0\right]\psi_{\text{II}}(x) = E\,\psi_{\text{II}}(x).$$

indicates that we are in "Region II" (see FIG 10).

Hence:

$$\text{(b)}\quad \left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}+V_0-E\right]\psi_{\text{II}}(x) = 0.$$

Now multiply both sides by $-2m/\hbar^2$...

$$\text{(c)}\quad \left[\frac{d^2}{dx^2}-\underbrace{\frac{2m}{\hbar^2}(V_0-E)}_{K^2,\ \text{from (93)}}\right]\psi_{\text{II}}(x)=0$$

Hence:

$$\text{(d)}\quad \left[\frac{d^2}{dx^2}-K^2\right]\psi_{\text{II}}(x)=0.$$

Now substitute expression (92), for $\psi_{\text{II}}(x)$, into the left side of (d) ... and show that we indeed get zero:

$$\text{(e)}\quad \left[\frac{d^2}{dx^2}-K^2\right]\left(Fe^{Kx}+Ge^{-Kx}\right)$$

$$= F\frac{d^2}{dx^2}e^{Kx}+G\frac{d^2}{dx^2}e^{-Kx}-K^2\left(Fe^{Kx}+Ge^{-Kx}\right)$$

$$= F(K^2)e^{Kx}+G(-K)^2e^{-Kx}-K^2\left(Fe^{Kx}+Ge^{-Kx}\right)$$

$$= 0,\ \text{as required.}$$

(b). From (94), and making use of (87)|(90)|(92), we have:

$$\begin{cases} (f)\ \psi_{I}(x=0)=\psi_{II}(x=0) \Rightarrow A+B=F+G \\ (g)\ \psi_{II}(x=a)=\psi_{III}(x=a) \Rightarrow Fe^{Ka}+Ge^{-Ka}=Ce^{ika}+De^{-ika} \end{cases}$$

Now, we look at (95), again making use of (87)|(90)|(92):

$$\begin{cases} (h)\ \psi_{I}'(x=0)=\psi_{II}'(x=0) \Rightarrow A(ik)+B(-ik)=FK-GK \\ (i)\ \psi_{II}'(x=a)=\psi_{III}'(x=a) \Rightarrow FKe^{Ka}-GKe^{-Ka}=ikCe^{ika}-ikDe^{-ika}. \end{cases}$$

Now, in (b), we want to obtain an expression for:

$$(j)\quad R \overset{(91a)}{=} |B|^2/|A|^2.$$

Hence we need to solve equations (f)|(g)|(h)|(i) for B/A.

Begin by writing (f) through (i) in a "clearer" form:

$$(k)\begin{cases} (f)\ A+B=F+G \\ (g)\ Ce^{ika}=Fe^{Ka}+Ge^{-Ka} \\ (h)\ ik(A-B)=K(F-G) \\ (i)\ ikCe^{ika}=K(Fe^{Ka}-Ge^{-Ka}) \end{cases}$$

here, we have set $D=0$ (see FIG 10) (see footnote 34)

Eliminate F and G from (k), then solve for $\frac{B}{A}$ to see:

$$(l)\quad \frac{B}{A}=\frac{(k^2+K^2)(e^{2Ka}-1)}{e^{2Ka}(k+iK)^2-(k-iK)^2}.$$

Taking the squared modulus, of the above equation, and making use of (j), leads to the required result – namely eq. (96) of the notes.

(c) Eliminate F and G from (k), then solve for C/A, so:

$$(m)\quad \frac{C}{A}=\frac{4ikKe^{-ika}e^{Ka}}{e^{2Ka}(k+iK)^2-(k-iK)^2}$$

The squared modulus of (m) may then be substituted into (91b), yielding equation (97), as required.

(d) From (96) and (97),

$$R+T = \left[1+\frac{4E(V_0-E)}{V_0^2\sinh^2(Ka)}\right]^{-1} + \left[1+\frac{V_0^2\sinh^2(Ka)}{4E(V_0-E)}\right]^{-1} \quad (n)$$

Let: (p) $\Theta = \dfrac{4E(V_0-E)}{V_0^2\sinh^2(Ka)}$, so that (n) becomes:

$$(q)\quad R+T = [1+\Theta]^{-1} + [1+\Theta^{-1}]^{-1}$$

$$= \frac{1}{1+\Theta} + \frac{1}{1+\Theta^{-1}}$$

$$= \frac{1}{1+\Theta} + \frac{\Theta}{\Theta}\left(\frac{1}{1+\Theta^{-1}}\right) = \frac{1}{1+\Theta} + \frac{\Theta}{\Theta+1}$$

$$= \frac{1+\Theta}{1+\Theta}$$

$= 1$, as required

Exercise #30 Solution very similar to #29 !! ☺

Exercise #31 (a) In the region $-a < x < a$, the potential is zero and so the required time-independent Schrödinger equation is (86), namely:

$$(a)\quad -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n(x) = E_n\psi_n(x).$$

Substitute (102) into the left side of (a), hence:

$$(b)\quad -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\frac{1}{\sqrt{a}}\begin{Bmatrix}\cos \to n \text{ odd}\\ \sin \to n \text{ even}\end{Bmatrix}\left(\frac{n\pi x}{2a}\right)$$

$$= -\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{a}}\right)(-1)\left(\frac{n\pi}{2a}\right)^2\begin{Bmatrix}\cos \to n \text{ odd}\\ \sin \to n \text{ even}\end{Bmatrix}\left(\frac{n\pi x}{2a}\right)$$

$$= -\frac{\hbar^2}{2m}(-1)\left(\frac{n\pi}{2a}\right)^2\psi_n(x)$$

$$= \underbrace{\frac{\hbar^2\pi^2 n^2}{8ma^2}}_{\text{from (103), this is } E_n}\psi_n(x)$$

$= E_n\psi_n(x)$, as required.

(b) Case where n = odd

$$\int_{-a}^{a} |\psi_n(x)|^2 dx \overset{(102)}{=} \frac{1}{a}\int_{-a}^{a} \cos^2\left(\frac{n\pi x}{2a}\right) dx$$

[recall $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$]

$$= \frac{1}{a}\int_{-a}^{a}\frac{1}{2}\left\{1+\underbrace{\cos\left(\frac{n\pi x}{a}\right)}\right\} dx$$

period is $\frac{2\pi}{n\pi/a} = \frac{2a}{n}$, hence there are an integer number of periods in the integration domain, hence the integral is zero.

$$= \frac{1}{2a}\int_{-a}^{a} 1\, dx = \frac{1}{2a} \times 2a = 1$$

Case where n = even

$$\int_{-a}^{a} |\psi_n(x)|^2 dx \overset{(102)}{=} \frac{1}{a}\int_{-a}^{a} \sin^2\left(\frac{n\pi x}{2a}\right) dx$$

[recall $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$]

$$= \frac{1}{a}\int_{-a}^{a}\frac{1}{2}\left\{1-\underbrace{\cos\left(\frac{n\pi x}{a}\right)}\right\} dx$$

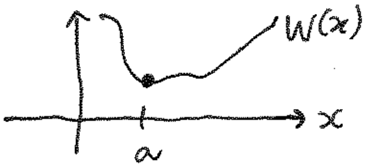
$$= \frac{1}{a}\int_{-a}^{a}\frac{1}{2}\, dx$$

$$= \frac{1}{2a} \times 2a = 1$$

(c) When $x > a$ or $x < -a$, $\psi(x)$ is equal to zero. Hence we need to show that $\psi(x=-a) = \psi(x=a) = 0$, in order to see that ψ is continuous at the edges of the square well. I'll leave this to you!

(d) We have already answered this, in the course of (a)!

Exercise #32 Taylor expand $W(x)$ about $x = a$ (cf. FIG 13):



(a) $W(x) = W(x=a) + W'(x=a)(x-a) + W''(x=a)\frac{1}{2!}(x-a)^2 + \ldots$

Now, $W'(x=a) = 0$, since there is a local minimum of $W(x)$ at $x = a$. Hence (a) becomes:

(b) $W(x) \approx W(x=a) + W''(x-a)\frac{1}{2!}(x-a)^2$

← higher-order terms have been omitted

(c) $W(x) - \underbrace{W(x=a)}_{\text{this is a constant; call it } W_0} = \overbrace{W''(x-a)\frac{1}{2!}}^{\text{this is a constant; call it } \frac{1}{2}k}(x-a)^2$

(d) $W(x) - W_0 = \frac{1}{2}k(x-a)^2$.

Now, we can put the x origin anywhere we like. Let us put the x origin at a, so that we can set $a = 0$ in (d), thus:

(e) $$W(x) - W_0 = \frac{1}{2}kx^2$$

Also, we can choose the zero of potential at will; the potential $W(x) - W_0$ can thus be relabelled $V(x)$. Equation (e) therefore becomes (104), as required.

Exercise #33 Let's do this in reverse, by showing that (107) is the same as (105). Now, (107) is:

(a) $$\frac{d^2}{d\xi^2}\psi(\xi) + (\alpha - \xi^2)\psi(\xi) = 0.$$

From (106), we have:

(b) $$\frac{d^2}{d(\alpha x)^2}\psi + \left(\frac{2E}{\hbar\omega} - \alpha^2 x^2\right)\psi = 0$$

(c) $$\frac{1}{\alpha^2}\frac{d^2}{dx^2}\psi + \left(\frac{2E}{\hbar\omega} - \alpha^2 x^2\right)\psi = 0$$

Now, $$\alpha^2 \overset{(106d)}{=} \frac{m\omega}{\hbar} = \frac{m}{\hbar}\{\omega\} \overset{(106b)}{=} \frac{m}{\hbar}\sqrt{\frac{k}{m}} = \frac{\sqrt{km}}{\hbar}$$

so that (c) becomes:

(e) $$\frac{\hbar}{\sqrt{km}}\frac{d^2}{dx^2}\psi + \left(\frac{2E}{\hbar\underbrace{\sqrt{k/m}}} - \frac{\sqrt{km}}{\hbar}x^2\right)\psi = 0$$

note we used eq. (106b) for ω

Now multiply through by $-\frac{\hbar\sqrt{k}}{2\sqrt{m}}$, so that:

(f) $$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi - \frac{\hbar\sqrt{k}}{2\sqrt{m}}\left(\frac{2E}{\hbar}\sqrt{\frac{m}{k}} - \frac{\sqrt{km}}{\hbar}x^2\right)\psi = 0$$

(g) $$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi - E\psi + \frac{1}{2}kx^2\psi = 0.$$ This is the same as (105), as required.

Exercise # 34

(a) $$\frac{d^2}{d\xi^2}\psi(\xi) \overset{(109)}{=} \frac{d^2}{d\xi^2}\xi^p e^{-\frac{1}{2}\xi^2}$$

$$= \frac{d}{d\xi}\Big(\underbrace{\frac{d}{d\xi}\xi^p e^{-\frac{1}{2}\xi^2}}_{\text{use product rule}}\Big)$$

$$= \frac{d}{d\xi}\left(\xi^p \frac{d}{d\xi}e^{-\frac{1}{2}\xi^2} + e^{-\frac{1}{2}\xi^2}\frac{d}{d\xi}\xi^p\right)$$

$$= \frac{d}{d\xi}\left(\xi^p(-\xi)e^{-\frac{1}{2}\xi^2} + e^{-\frac{1}{2}\xi^2}p\xi^{p-1}\right)$$

$$= \frac{d}{d\xi}\left(-\xi^{p+1}e^{-\frac{1}{2}\xi^2} + p\xi^{p-1}e^{-\frac{1}{2}\xi^2}\right)$$ ... use product rule again...

$$= (-\xi^{p+1})\frac{d}{d\xi}e^{-\frac{1}{2}\xi^2} + e^{-\frac{1}{2}\xi^2}\frac{d}{d\xi}(-\xi^{p+1}) + p\xi^{p-1}\frac{d}{d\xi}e^{-\frac{1}{2}\xi^2}$$
$$+ e^{-\frac{1}{2}\xi^2}\frac{d}{d\xi}(p\xi^{p-1})$$

$$= (-\xi^{p+1})(-\xi)e^{-\frac{1}{2}\xi^2} + e^{-\frac{1}{2}\xi^2}(-(p+1))\xi^p$$
$$+ p\xi^{p-1}(-\xi)e^{-\frac{1}{2}\xi^2} + e^{-\frac{1}{2}\xi^2}p(p-1)\xi^{p-2}$$

$$= \exp(-\tfrac{1}{2}\xi^2)\left\{\xi^{p+2} - (p+1)\xi^p - p\xi^p + p(p-1)\xi^{p-2}\right\}$$

since $|\xi|$ is large, all lower powers (ξ^{p+1}, ξ^p, ξ^{p-1}, ξ^{p-2} etc.) can be ignored

$$\approx \exp(-\tfrac{1}{2}\xi^2)\xi^{p+2}$$

$$= \xi^2\exp(-\tfrac{1}{2}\xi)^2\xi^p$$

$$\overset{(109)}{=} \xi^2\psi(\xi), \quad |\xi|\to\infty,$$ as required.

(b) The mathematics is similar to that above, and will not be given here. We reject the solution $\psi(\xi) = \xi^p\exp(+\xi^2/2)$, because it diverges as $|\xi|\to\infty$, and is therefore unphysical.

Exercise #35 Substitute (110) into (107), hence:

$$\frac{d^2}{d\xi^2}\psi(\xi)+(\lambda-\xi^2)\psi(\xi)=0$$

$$\text{(a)}\quad \underbrace{\frac{d^2}{d\xi^2}\left[e^{-\frac{1}{2}\xi^2}H(\xi)\right]}+(\lambda-\xi^2)e^{-\frac{1}{2}\xi^2}H(\xi)=0$$

we focus on this term, for the moment:

$$\text{(b)}\quad \frac{d^2}{d\xi^2}\left[e^{-\frac{1}{2}\xi^2}H(\xi)\right]$$

$$=\frac{d}{d\xi}\left(\underbrace{\frac{d}{d\xi}\left(e^{-\frac{1}{2}\xi^2}H(\xi)\right)}\right)$$

now use product rule

$$=\frac{d}{d\xi}\left[e^{-\frac{1}{2}\xi^2}\frac{d}{d\xi}H(\xi)+H(\xi)\frac{d}{d\xi}e^{-\frac{1}{2}\xi^2}\right]$$

$$=\frac{d}{d\xi}\left[e^{-\frac{1}{2}\xi^2}\frac{d}{d\xi}H(\xi)-\xi H(\xi)e^{-\frac{1}{2}\xi^2}\right]$$

... now use product rule again....

$$=e^{-\frac{1}{2}\xi^2}\frac{d^2}{d\xi^2}H(\xi)+\left(\frac{d}{d\xi}e^{-\frac{1}{2}\xi^2}\right)\left(\frac{d}{d\xi}H(\xi)\right)$$
$$-\xi H(\xi)\frac{d}{d\xi}\left(e^{-\frac{1}{2}\xi^2}\right)-e^{-\frac{1}{2}\xi^2}\frac{d}{d\xi}\left(\xi H(\xi)\right)$$

$$=e^{-\frac{1}{2}\xi^2}\frac{d^2}{d\xi^2}H(\xi)-\xi e^{-\frac{1}{2}\xi^2}\frac{d}{d\xi}H(\xi)$$
$$+\xi^2H(\xi)e^{-\frac{1}{2}\xi^2}-e^{-\frac{1}{2}\xi^2}\xi\frac{d}{d\xi}H(\xi)$$
$$-e^{-\frac{1}{2}\xi^2}H(\xi)$$

product rule yet again!

... now factor out $\exp(-\frac{1}{2}\xi^2)$ ... also, let $\frac{d}{d\xi}H(\xi)\equiv H'$, and $\frac{d^2}{d\xi^2}H(\xi)\equiv H''$...

$$=e^{-\frac{1}{2}\xi^2}\left[H''-\xi H'+\xi^2H-\xi H'-H\right]$$

Making use of (b), (a) becomes:

$$\text{(c)}\quad \cancel{e^{-\frac{1}{2}\xi^2}}\left[H''-\xi H'+\cancel{\xi^2}H-\xi H'-H+\lambda H-\cancel{\xi^2 H}\right]=0$$

Hence: $H''-2\xi H'+(\lambda-1)H=0$, as required.

Exercise #36 Begin with the Hermite equation (111), and then substitute in (113). Thus (111) becomes:

$$H'' - 2\xi H' + (\lambda-1)H = 0$$

$$\frac{d^2}{d\xi^2}\sum_{k=0}^{\infty} c_k \xi^{2k} - 2\xi \frac{d}{d\xi}\sum_{k=0}^{\infty} c_k \xi^{2k} + (\lambda-1)\sum_{k=0}^{\infty} c_k \xi^{2k} = 0$$

(a) $$\underbrace{\sum_{k=1}^{\infty} c_k\, 2k(2k-1)\xi^{2k-2}}_{\text{let } 2k' \equiv 2k-2 \;\Rightarrow\; k' \equiv k-1 \text{ and } k = k'+1} - 2\xi\sum_{k=1}^{\infty} c_k (2k)\xi^{2k-1} + (\lambda-1)\sum_{k=0}^{\infty} c_k \xi^{2k} = 0$$

$k=1$ ← note! $k=1$ ← note!

absorb into summation

Hence:

(b) $$\sum_{k'=0}^{\infty} c_{k'+1}\, 2(k'+1)(2k'+1)\xi^{2k'} - 2\sum_{k=1}^{\infty} c_k (2k)\xi^{2k} + (\lambda-1)\sum_{k=0}^{\infty} c_k \xi^{2k} = 0$$

← note! (when $k=1$, $k'=k-1=1-1=0$)

Now drop the primes from the first summation in (b). We can then combine together all of the summations that appear in (b), hence:

(c) $$\sum_{k=0}^{\infty} \xi^{2k}\left\{ c_{k+1}\, 2(k+1)(2k+1) - 2c_k(2k) + (\lambda-1)c_k \right\} = 0$$

Hence:

(d) $$\sum_{k=0}^{\infty} \xi^{2k}\left\{ 2(k+1)(2k+1)c_{k+1} - (4k+1-\lambda)c_k \right\} = 0,$$

as required.

Exercise #37 The Taylor series expansion for e^x is:

(a) $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

Hence the Taylor expansion for e^{ξ^2} is:

(b) $$\exp(\xi^2) = 1 + (\xi^2) + \frac{1}{2!}(\xi^2)^2 + \frac{1}{3!}(\xi^2)^3 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}(\xi^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!}\xi^{2k}$$

Hence,

(c) $$\xi^{2p}\exp(\xi^2) = \sum_{k=0}^{\infty} \frac{1}{k!}\xi^{2(k+p)}.$$

Let $k' = k + p$, hence $k = k' - p$ and (c) becomes:

(d) $$\xi^{2p}\exp(\xi^2) = \sum_{k'=p}^{\infty} \frac{1}{(k'-p)!}\xi^{2k'}.$$

Now drop the primes, and let:

(e) $$d_k = \frac{1}{(k-p)!},$$ so that (d) becomes:

(f) $$\xi^{2p}\exp(\xi^2) = \sum_{k=p}^{\infty} d_k \xi^{2k}. \quad (\text{cf } 113)$$

Let us calculate the ratio of successive coefficients in the above series (cf. 116):

(g) $$\frac{d_{k+1}}{d_k} \overset{(e)}{=} \frac{(k-p)!}{(k+1-p)!} = \frac{\cancel{(k-p)!}}{(k+1-p)\cancel{(k-p)!}}$$

$$= \frac{1}{(k+1-p)} \longrightarrow \frac{1}{k} \text{ for large } k.$$

The behaviour of $\xi^{2p}\exp(\xi^2)$ is therefore the same as in 116, for large k.

Exercise #38 Keeping only odd powers of ξ, we have (cf. 113):

(a) $$H(\xi) = \sum_{k=0}^{\infty} d_k \xi^{2k+1}.$$ Put this into 111. Using similar manipulations to those which led to 115, one thereby obtains the recursion relation:

(b) $$d_{k+1} = \frac{4k+3-\lambda}{2(k+1)(2k+3)} d_k.$$

Hence,

(c) $$\frac{d_{k+1}}{d_k} = \frac{4k+3-\lambda}{2(k+1)(2k+3)} \longrightarrow \frac{1}{k} \text{ for large } k \quad (\text{cf } 116).$$

We therefore have the same "blowup" behaviour as in (117), unless the series in (a) terminates after a finite number of terms. This is easily arranged by demanding that, for some value of k (call this "N"), the numerator in (c) should vanish. Thus:

(d) $4N+3-\lambda=0$ → From (106a), $\lambda=\frac{2E}{\hbar\omega}$

Hence: $$4N+3-\frac{2E}{\hbar\omega}=0$$

$$\Rightarrow E=\hbar\omega\left(2N+\frac{3}{2}\right)=\hbar\omega\left(2N+1+\frac{1}{2}\right),$$

$N=0,1,2,\ldots$ as required.

<u>Exercise #39</u> For the case of zero potential, (83) becomes:

(a) $$i\hbar\frac{\partial}{\partial t}\Psi=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi.$$

We now want to show that (122) obeys (a). We separately calculate the left side of (a) & the right side of (a), using (122). We shall see that the resulting expressions are equal, as required.

<u>Left side</u> Substitute (122) into the left side of (a), hence:

$$i\hbar\frac{\partial}{\partial t}\Psi=i\hbar\frac{\partial}{\partial t}\left\{\frac{1}{\pi^{1/4}}\sqrt{\frac{\Delta p_x/\hbar}{1+\frac{i(\Delta p_x)^2t}{m\hbar}}}\exp\left[\frac{\frac{ip_0x}{\hbar}-\left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2}+\frac{ip_0^2t}{2m\hbar}}{1+\frac{i(\Delta p_x)^2t}{m\hbar}}\right]\right\}$$

... now use product rule

(b) $$=\frac{i\hbar}{\pi^{1/4}}\sqrt{\frac{\Delta p_x/\hbar}{1+i(\Delta p_x)^2t/(m\hbar)}}\frac{\partial}{\partial t}\left\{\exp\left[\frac{\frac{ip_0x}{\hbar}-\left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2}+\frac{ip_0^2t}{2m\hbar}}{1+\frac{i(\Delta p_x)^2t}{m\hbar}}\right]\right\}$$

$$+\frac{i\hbar}{\pi^{1/4}}\exp\left[\frac{\frac{ip_0x}{\hbar}-\left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2}+\frac{ip_0^2t}{2m\hbar}}{1+\frac{i(\Delta p_x)^2t}{m\hbar}}\right]\frac{\partial}{\partial t}\sqrt{\frac{\Delta p_x/\hbar}{1+\frac{i(\Delta p_x)^2t}{m\hbar}}}$$

Making use of the chain and quotient rules, after some algebra you should end up with:

ⓒ $$i\hbar\frac{\partial}{\partial t}\Psi = \frac{i\sqrt{\Delta p_x}}{m\pi^{1/4}\sqrt{\hbar}}\exp\left[\frac{\frac{ip_0x}{\hbar}-\left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2}+\frac{ip_0^2t}{2m\hbar}}{1+\frac{i(\Delta p_x)^2t}{m\hbar}}\right]$$

$$\times\left\{\frac{-\frac{1}{2}ip_0^2+\frac{1}{\hbar}(\Delta p_x)^2p_0x+\frac{i}{2\hbar^2}(\Delta p_x)^4x^2}{\left(1+\frac{i(\Delta p_x)^2t}{m\hbar}\right)^{5/2}}-\frac{i(\Delta p_x)^2}{2\sqrt{1+\frac{i(\Delta p_x)^2t}{m\hbar}}^{\,3}}\right\}$$

Now for the right side of ⓐ:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi \overset{(22)}{=} Q\frac{\partial^2}{\partial x^2}\exp\left[A\left\{\frac{ip_0x}{\hbar}-\left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2}+\frac{ip_0^2t}{2m\hbar}\right\}\right]$$
ⓓ

where:

ⓔ $$Q\equiv-\frac{\hbar^2}{2m}\left(\frac{1}{\pi^{1/4}}\right)\sqrt{\frac{\Delta p_x/\hbar}{1+\frac{i(\Delta p_x)^2t}{m\hbar}}},$$ and

ⓕ $$A\equiv\left[1+\frac{i(\Delta p_x)^2t}{m\hbar}\right]^{-1}.$$ Now perform the differentiation in ⓓ, making use of the chain and product rules, as appropriate.

After some algebra, you will end up with:

ⓖ $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi=\frac{QA^2}{\hbar^2}e^{III}\left\{\left(ip_0-\frac{(\Delta p_x)^2x}{\hbar}\right)^2-\frac{(\Delta p_x)^2}{A}\right\}$$

where:

ⓗ $$III\equiv\left(\frac{ip_0x}{\hbar}-\left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2}+\frac{ip_0^2t}{2m\hbar}\right)\Big/\left(1+\frac{i(\Delta p_x)^2t}{m\hbar}\right).$$

Using the same "shorthand" defined in ⓔ / ⓕ / ⓗ, eq. ⓒ becomes:

ⓘ $$i\hbar\frac{\partial}{\partial t}\Psi=\frac{i\sqrt{\Delta p_x/\hbar}}{m\pi^{1/4}}e^{III}\times\frac{1}{\sqrt{1+\frac{i(\Delta p_x)^2t}{m\hbar}}}\left\{\frac{-\frac{1}{2}ip_0^2+\frac{1}{\hbar}(\Delta p_x)^2p_0x+\frac{i}{2\hbar^2}(\Delta p_x)^4x^2}{A^{-2}}-\frac{i(\Delta p_x)^2}{2A^{-1}}\right\}$$

$$=e^{III}\times\frac{i}{\cancel{m\pi^{1/4}}}\times\frac{-2\cancel{m}}{\hbar^2}\times\cancel{\pi^{1/4}}\times Q\times A^2\left\{-\frac{1}{2}ip_0^2+\frac{1}{\hbar}(\Delta p_x)^2p_0x+\frac{i}{2\hbar^2}(\Delta p_x)^4x^2-\frac{i(\Delta p_x)^2}{2A}\right\}$$

$$=\frac{QA^2}{\hbar^2}\times\underbrace{(-2i)}_{\text{multiply through}}\times e^{III}\times\left\{-\frac{1}{2}ip_0^2+\frac{1}{\hbar}(\Delta p_x)^2p_0x+\frac{i}{2\hbar^2}(\Delta p_x)^4x^2-\frac{i(\Delta p_x)^2}{2A}\right\}$$

$$= \frac{QA^2}{\hbar^2} e^{\cdots} \left\{ -p_0^2 - \frac{2i}{\hbar}(\Delta p_x)^2 p_0 x + \frac{1}{\hbar^2}(\Delta p_x)^4 x^2 - \frac{(\Delta p_x)^2}{A} \right\}$$

$$= \frac{QA^2}{\hbar^2} e^{\cdots} \left\{ \left(ip_0 - \frac{(\Delta p_x)^2 x}{\hbar} \right)^2 - \frac{(\Delta p_x)^2}{A} \right\},$$

which is the same as the right side of (9) — QED!

<u>Exercise #40</u> $\rho(x,t) = |\Psi(x,t)|^2$

$$= \left| \frac{1}{\pi^{1/4}} \sqrt{\frac{\Delta p_x/\hbar}{1 + \frac{i(\Delta p_x)^2 t}{m\hbar}}} \exp\left[\frac{\frac{ip_0 x}{\hbar} - \left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{x^2}{2} + \frac{ip_0^2 t}{2m\hbar}}{1 + \frac{i(\Delta p_x)^2 t}{m\hbar}} \right] \right|^2$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{\Delta p_x}{\hbar} \times \frac{1}{\sqrt{\left(1 + \frac{i(\Delta p_x)^2 t}{m\hbar}\right)\left(1 - \frac{i(\Delta p_x)^2 t}{m\hbar}\right)}}$$

$$\times \exp\left[\frac{\frac{ip_0 x}{\hbar} - \left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{x^2}{2} + \frac{ip_0^2 t}{2m\hbar}}{1 + \frac{i(\Delta p_x)^2 t}{m\hbar}} \right] \exp\left[\frac{-\frac{ip_0 x}{\hbar} - \left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{x^2}{2} - \frac{ip_0^2 t}{2m\hbar}}{1 - \frac{i(\Delta p_x)^2 t}{m\hbar}} \right]$$

note!

$$= \frac{1}{\sqrt{\pi}} \times \frac{\Delta p_x/\hbar}{\sqrt{1 + \frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}}$$

$$\times \exp\left[\frac{\frac{ip_0 x}{\hbar} - \left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{x^2}{2} + \frac{ip_0^2 t}{2m\hbar}}{1 + \frac{i(\Delta p_x)^2 t}{m\hbar}} - \frac{\frac{ip_0 x}{\hbar} + \left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{x^2}{2} + \frac{ip_0^2 t}{2m\hbar}}{1 - \frac{i(\Delta p_x)^2 t}{m\hbar}} \right]$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{\Delta p_x/\hbar}{\sqrt{1 + \frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}}$$

$$\times \exp\left[\frac{\left[\frac{ip_0 x}{\hbar} - \left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{x^2}{2} + \frac{ip_0^2 t}{2m\hbar}\right]\left[1 - \frac{i(\Delta p_x)^2 t}{m\hbar}\right] - \left[\frac{ip_0 x}{\hbar} + \left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{x^2}{2} + \frac{ip_0^2 t}{2m\hbar}\right]\left[1 + \frac{i(\Delta p_x)^2 t}{m\hbar}\right]}{1 + \frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}} \right]$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{\Delta p_x/\hbar}{\sqrt{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}}$$

$$\times \exp\left[\frac{\left\{\cancel{\frac{i p_0 x}{\hbar}} - \left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2} + \cancel{\frac{i p_0^2 t}{2m\hbar}} + \frac{p_0 x (\Delta p_x)^2 t}{\hbar^2 m} + \cancel{\frac{(\Delta p_x)^4 x^2 i t}{2\hbar^3 m}} + \frac{p_0^2 t^2 (\Delta p_x)^2}{2m^2\hbar^2}\right\} - \left[\cancel{\frac{i p_0 x}{\hbar}} + \left(\frac{\Delta p_x}{\hbar}\right)^2\frac{x^2}{2} + \cancel{\frac{i p_0^2 t}{2m\hbar}} - \frac{p_0 x (\Delta p_x)^2 t}{\hbar^2 m} + \cancel{\frac{(\Delta p_x)^4 x^2 i t}{\hbar^3 2m}} - \frac{p_0^2 t^2 (\Delta p_x)^2}{2m^2\hbar^2}\right]}{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}\right]$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{\Delta p_x/\hbar}{\sqrt{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}} \times \exp\left[\frac{\left(\frac{\Delta p_x}{\hbar}\right)^2\left\{-x^2 + \frac{2p_0 x t}{m} + \frac{p_0^2 t^2}{m^2}\right\}}{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}\right]$$

[Re: term in braces — let $v = p_0/m$]

$$= \frac{1}{\sqrt{\pi}} \frac{\Delta p_x/\hbar}{\sqrt{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}} \exp\left[\frac{\left(\frac{\Delta p_x}{\hbar}\right)^2\left\{-x^2 + 2vxt + v^2 t^2\right\}}{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}\right]$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Delta p_x/\hbar}{\sqrt{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}} \exp\left[-\frac{\left(\frac{\Delta p_x}{\hbar}\right)^2\left\{x - vt\right\}^2}{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}\right]$$

This should be negative — I don't know how to fix it! ☹ Please let me know if you know how to fix this!

Exercise #41 Freeze time at a given t. Then, from (123), the peak value of the probability density will be:

(a) $$\rho_{peak}(t) = \frac{1}{\sqrt{\pi}} \frac{\Delta p_x/\hbar}{\sqrt{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}}.$$

The positions x, at which $\rho = \frac{1}{e}\rho_{peak}$, will obey:

(b) $$\frac{1}{e}\frac{1}{\sqrt{\pi}} \frac{\Delta p_x/\hbar}{\sqrt{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}} = \frac{1}{\sqrt{\pi}} \frac{\Delta p_x/\hbar}{\sqrt{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}} \exp\left[-\frac{\left(\frac{\Delta p_x}{\hbar}\right)^2 (x-vt)^2}{1+\frac{(\Delta p_x)^4 t^2}{m^2\hbar^2}}\right]$$

Hence,

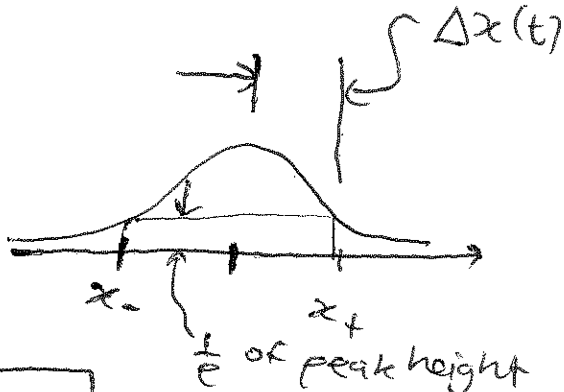
(c) $$e^{-1} = \exp\left[- \frac{\left(\frac{\Delta p_x}{\hbar}\right)^2 (x - vt)^2}{\underbrace{1 + \frac{(\Delta p_x)^4 t^2}{m^2 \hbar^2}}} \right]$$

Thus, this exponent equals -1.

$$\Rightarrow -1 = -\left(\frac{\Delta p_x}{\hbar}\right)^2 \frac{(x - vt)^2}{1 + \frac{(\Delta p_x)^4 t^2}{m^2 \hbar^2}}$$

$$\Rightarrow \frac{\hbar}{\Delta p_x}\sqrt{1 + \frac{(\Delta p_x)^4 t^2}{m^2 \hbar^2}} = x - vt$$

$$\Rightarrow x_{\pm} = vt \pm \frac{\hbar}{\Delta p_x}\sqrt{1 + \frac{(\Delta p_x)^4 t^2}{m^2 \hbar^2}}$$



Hence,

$$\Delta x(t) = \frac{\hbar}{\Delta p_x}\sqrt{1 + \frac{(\Delta p_x)^4 t^2}{m^2 \hbar^2}},$$

as required.

Exercise #42

$$\langle \Psi_1 | \Psi_2 \rangle \equiv \iiint \Psi_1^* \Psi_2 \, d\vec{r}$$
$$= \left[\iiint \Psi_1 \Psi_2^* \, d\vec{r} \right]^*$$
$$= \left[\iiint \Psi_2^* \Psi_1 \, d\vec{r} \right]^*$$
$$\equiv \langle \Psi_2 | \Psi_1 \rangle^* \quad \text{QED}$$

$$\langle c\Psi_1 | \Psi_2 \rangle \equiv \iiint (c\Psi_1)^* \Psi_2 \, d\vec{r}$$
$$= \iiint c^* \Psi_1^* \Psi_2 \, d\vec{r}$$
$$= c^* \iiint \Psi_1^* \Psi_2 \, d\vec{r}$$
$$\equiv c^* \langle \Psi_1 | \Psi_2 \rangle \quad \text{QED}$$

$\langle \Psi_3 | \Psi_1 + \Psi_2 \rangle$

$$\equiv \iiint \Psi_3^* (\Psi_1 + \Psi_2) d\vec{r}$$

$$= \iiint \Psi_3^* \Psi_1 d\vec{r} + \iiint \Psi_3^* \Psi_2 d\vec{r}$$

$$\equiv \langle \Psi_3 | \Psi_1 \rangle + \langle \Psi_3 | \Psi_2 \rangle \quad \text{QED}$$

Exercise #43 Take (135), letting $\Psi = c_1\Psi_1 + c_2\Psi_2$, so:

$$\langle c_1\Psi_1 + c_2\Psi_2 | A(c_1\Psi_1 + c_2\Psi_2) \rangle$$

$$= \langle A(c_1\Psi_1 + c_2\Psi_2) | c_1\Psi_1 + c_2\Psi_2 \rangle$$

Now recall (130) and (131), allowing us to proceed:

$$\cancel{c_1^* c_1 \langle \Psi_1 | A\Psi_1 \rangle} + c_1^* c_2 \langle \Psi_1 | A\Psi_2 \rangle + c_2^* c_1 \langle \Psi_2 | A\Psi_1 \rangle$$

$$\cancel{+ c_2^* c_2 \langle \Psi_2 | A\Psi_2 \rangle} = \cancel{c_1^* c_1 \langle A\Psi_1 | \Psi_1 \rangle} + c_1^* c_2 \langle A\Psi_1 | \Psi_2 \rangle$$

$$+ c_2^* c_1 \langle A\Psi_2 | \Psi_1 \rangle \cancel{+ c_2^* c_2 \langle \Psi_2 | A\Psi_2 \rangle}$$

Note: in above cancellations, we used the facts that

$\langle \Psi_1 | A\Psi_1 \rangle = \langle A\Psi_1 | \Psi_1 \rangle$ &

$\langle \Psi_2 | A\Psi_2 \rangle = \langle A\Psi_2 | \Psi_2 \rangle$.

Hence:

$$c_2^* c_1 \left[\langle \Psi_2 | A\Psi_1 \rangle - \langle A\Psi_2 | \Psi_1 \rangle \right]$$

$$+ c_1^* c_2 \left[\langle \Psi_1 | A\Psi_2 \rangle - \langle A\Psi_1 | \Psi_2 \rangle \right] = 0$$

Now, the complex numbers c_1 & c_2 are arbitrary. Also, $c_2^* c_1$ and $c_1^* c_2$ — being complex conjugates — are linearly independent. Hence each quantity in square brackets must vanish. This yields the required result.

Exercise #44 Start with (138), and then multiply both sides by c^*:

$$c^*\langle \Psi_1 | A^\dagger \Psi_2\rangle = c^*\langle A\Psi_1 | \Psi_2\rangle$$

$$\Rightarrow \langle \Psi_1 | \underbrace{A^\dagger c^*}_{\text{this is the Hermitian conjugate of } cA} \Psi_2\rangle = \langle (cA)\Psi_1 | \Psi_2\rangle$$

$$\Rightarrow (cA)^\dagger = c^* A^\dagger \quad \text{QED}$$

$$\langle \Psi_1 | B^\dagger A^\dagger \Psi_2\rangle = \langle \Psi_1 | B^\dagger (A^\dagger \Psi_2)\rangle$$

$$= \langle B\Psi_1 | (A^\dagger \Psi_2)\rangle$$

$$= \langle B\Psi_1 | A^\dagger \Psi_2\rangle$$

$$= \langle AB\Psi_1 | \Psi_2\rangle$$

Hence this is indeed the Hermitian conjugate of AB, as required.

Exercise #45

$$\Psi = \sum_n \langle \psi_n | \Psi\rangle \psi_n$$

$$= \sum_n \iiint \psi_n^*(\vec{r}\,')\Psi(\vec{r}\,')d\vec{r}\,'\,\psi_n(\vec{r})$$

$$= \iiint \Psi(\vec{r}\,') \underbrace{\sum_n \psi_n^*(\vec{r}\,')\psi_n(\vec{r})}_{\delta(\vec{r}-\vec{r}\,') \text{ as required}} d\vec{r}\,'$$

$$= \iiint \Psi(\vec{r}\,')\delta(\vec{r}-\vec{r}\,')d\vec{r}\,'$$

$$= \Psi(\vec{r})$$

Exercise #46 (a) Since A is Hermitian,

(a) $\langle \Psi_1 | A \Psi_2 \rangle = \langle A \Psi_1 | \Psi_2 \rangle$.

Now, from (165b) we have (b) $\hat{A} = UAU^\dagger$.

Take (b), premultiply both sides by $U^\dagger$ and postmultiply " " " U. Hence:

(c) $U^\dagger \hat{A} U = \underbrace{U^\dagger U}_{} A \underbrace{U^\dagger U}_{}$ ← $U^\dagger U = UU^\dagger = I$, from (145)

Hence:

(d) $A = U^\dagger \hat{A} U$, which can be put into (a):

(e) $\langle \Psi_1 | U^\dagger \hat{A} U \Psi_2 \rangle = \langle U^\dagger \hat{A} U \Psi_1 | \Psi_2 \rangle$

$\langle U\Psi_1 | \hat{A} U \Psi_2 \rangle = \langle \hat{A} U \Psi_1 | U \Psi_2 \rangle$

Now use (161), thus:

$\langle \hat{\Psi}_1 | \hat{A} \hat{\Psi}_2 \rangle = \langle \hat{A} \hat{\Psi}_1 | \hat{\Psi}_2 \rangle$,

demonstrating that $\hat{A}$ is Hermitian

(b) $[A, B] = C$

$AB - BA = C$ From (d), $A = U^\dagger \hat{A} U$ and $B = U^\dagger \hat{B} U$

$\Rightarrow U^\dagger \hat{A} \underbrace{UU^\dagger}_{} \hat{B} U - U^\dagger \hat{B} \underbrace{UU^\dagger}_{} \hat{A} U = C$

$\Rightarrow U^\dagger \hat{A} \hat{B} U - U^\dagger \hat{B} \hat{A} U = C$ now premultiply by U and postmultiply by $U^\dagger$

$\cancel{UU^\dagger} \hat{A} \hat{B} \cancel{UU^\dagger} - \cancel{UU^\dagger} \hat{B} \hat{A} \cancel{UU^\dagger} = C \cancel{UU^\dagger}$

$\Rightarrow \hat{A}\hat{B} - \hat{B}\hat{A} = C \Rightarrow [\hat{A}, \hat{B}] = 0$. QED!

(c) Let $\begin{cases} \lambda_n = \text{eigenvalues of } A, \\ \psi_n = \text{eigenfunctions of } A, \end{cases}$

$\Rightarrow A\psi_n = \lambda_n \psi_n$ ... now use (d)

$\Rightarrow \underbrace{U^\dagger \hat{A} U}\psi_n = \lambda_n \psi_n$

... now premultiply both sides by U

$\cancel{UU^\dagger} \hat{A} U\psi_n = \lambda_n U\psi_n$

$\hat{A}\hat{\psi}_n = \lambda_n \hat{\psi}_n \Rightarrow$ A and $\hat{A}$ have the same eigenvalues.

(d)

$$\langle \Phi | A\Psi \rangle = \langle U^\dagger \hat{\Phi} | \underbrace{U^\dagger \hat{A} U}_{\text{from (d)}} \underbrace{U^\dagger \hat{\Psi}}_{(162)} \rangle$$

(162)

$$= \langle U^\dagger \hat{\Phi} | U^\dagger \hat{A} \cancel{UU^\dagger} \hat{\Psi} \rangle$$

$$= \langle U^\dagger \hat{\Phi} | U^\dagger \hat{A} \hat{\Psi} \rangle$$

$$= \langle \cancel{UU^\dagger} \hat{\Phi} | \hat{A} \hat{\Psi} \rangle$$

$$= \langle \hat{\Phi} | \hat{A} \hat{\Psi} \rangle \quad \text{QED}$$

Exercise #47(a) (a) $U = 1 + i\varepsilon F$

(b) $U^\dagger = 1 - i\varepsilon F^\dagger$

Since U is unitary, $U^\dagger U = 1$ (see (145)). Thus:

(c) $1 = U^\dagger U$ ... now use (a) (b)

$= (1 + i\varepsilon F)(1 - i\varepsilon F^\dagger)$

$= 1 - i\varepsilon F^\dagger + i\varepsilon F + O(\varepsilon^2)$ "term(s) of order ε^2 or smaller"

Hence $1 = 1 + i\varepsilon(F - F^\dagger)$

$\Rightarrow F - F^\dagger = 0$

$\Rightarrow F = F^\dagger$

$\Rightarrow F$ is Hermitian. QED.

(b) From (161a), wavefunctions Ψ transform as:

$$\Psi \to \hat{\Psi} = U\Psi = (1 + i\varepsilon F)\Psi.$$

From (165b), operators transform as:

$$\begin{aligned} A \to \hat{A} &= UAU^\dagger \\ &= (1 + i\varepsilon F) A (1 - i\varepsilon F) \\ &= (A + i\varepsilon FA)(1 - i\varepsilon F) \\ &= A - i\varepsilon AF + i\varepsilon FA + O(\varepsilon^2) \\ &= A + i\varepsilon[F, A] + O(\varepsilon^2). \end{aligned}$$

Exercise #48 Let A be Hermitian. From (171), the matrix elements are:

(a)
$$\begin{aligned} A_{pm} &= \langle \psi_p | A \psi_m \rangle \\ &= \langle A\psi_p | \psi_m \rangle \quad \text{since } A \text{ is Hermitian} \\ &= \langle \psi_m | A \psi_p \rangle^* \\ &= A^*_{mp} \end{aligned}$$

Therefore the matrix, of matrix elements, is a Hermitian matrix, as required.

Exercise #49

$$[\hat{a}, \hat{a}^\dagger] \overset{(177)}{=} \left[\frac{m\omega\hat{q} + i\hat{p}}{\sqrt{2\hbar m\omega}}, \frac{m\omega\hat{q} - i\hat{p}}{\sqrt{2\hbar m\omega}} \right]$$

$$= \frac{1}{2\hbar m\omega}\left[m\omega\hat{q}+i\hat{p}, m\omega\hat{q}-i\hat{p}\right]$$

$$= \frac{1}{2\hbar m\omega}\left\{(m\omega\hat{q}+i\hat{p})(m\omega\hat{q}-i\hat{p})-(m\omega\hat{q}-i\hat{p})(m\omega\hat{q}+i\hat{p})\right\}$$

$$= \frac{1}{2\hbar m\omega}\left\{\cancel{m^2\omega^2\hat{q}^2}-im\omega\hat{q}\hat{p}+im\omega\hat{p}\hat{q}+\cancel{\hat{p}^2}-\cancel{m^2\omega^2\hat{q}^2}-im\omega\hat{q}\hat{p}+im\omega\hat{p}\hat{q}-\cancel{\hat{p}^2}\right\}$$

$$= \frac{im\omega}{2\hbar m\omega}\left\{-\hat{q}\hat{p}+\hat{p}\hat{q}-\hat{q}\hat{p}+\hat{p}\hat{q}\right\}$$

$$= \frac{im\omega}{\hbar m\omega}[\hat{p},\hat{q}] = \frac{im\omega}{\hbar m\omega}(-i\hbar) = -i^2 = 1 \quad \text{QED}$$

Exercise # 50

(175) $\Rightarrow \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2$

Now, one can solve (177) for $\hat{p}$ and $\hat{q}$; this gives:

(a) $\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)$; (b) $\hat{p} = \frac{\sqrt{2\hbar m\omega}}{2i}(\hat{a}-\hat{a}^\dagger)$,

which may be substituted into (175) to give:

(b) $$\hat{H} = \frac{1}{2m}\left(\frac{\sqrt{2\hbar m\omega}}{2i}\right)^2(\hat{a}-\hat{a}^\dagger)^2 + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)(\hat{a}+\hat{a}^\dagger)^2$$

$$= \frac{\hbar\omega}{4}\left\{-(\hat{a}-\hat{a}^\dagger)(\hat{a}-\hat{a}^\dagger) + (\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger)\right\}$$

$$= \frac{1}{4}\hbar\omega\left\{-\cancel{\hat{a}^2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}-\cancel{\hat{a}^{\dagger 2}}+\cancel{\hat{a}^2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\cancel{(\hat{a}^\dagger)^2}\right\}$$

$$= \frac{1}{4}\hbar\omega\left\{2\hat{a}\hat{a}^\dagger+2\hat{a}^\dagger\hat{a}\right\}$$

$$= \frac{1}{2}\hbar\omega\left\{\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right\}$$

From (178),
$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$
$\Rightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}+1$

$$= \frac{1}{2}\hbar\omega\left\{\hat{a}^\dagger\hat{a}+1+\hat{a}^\dagger\hat{a}\right\}$$

$$= \hbar\omega\left\{\hat{a}^\dagger\hat{a}+\frac{1}{2}\right\} \quad \text{QED}$$

Exercise #51

(a) $\hat{N}\hat{a}|\beta\rangle \overset{(180)}{=} \hat{a}^\dagger\hat{a}\hat{a}|\beta\rangle$

from (178), $\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$
$\Rightarrow \hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - 1$

$= (\hat{a}\hat{a}^\dagger - 1)\hat{a}|\beta\rangle$

$= \hat{a}\hat{a}^\dagger\hat{a}|\beta\rangle - \hat{a}|\beta\rangle$

$= \hat{a}\hat{N}|\beta\rangle - \hat{a}|\beta\rangle$ ... now use (182)

$= \hat{a}\beta|\beta\rangle - \hat{a}|\beta\rangle$

$= \beta\hat{a}|\beta\rangle - \hat{a}|\beta\rangle$

$= (\beta - 1)\hat{a}|\beta\rangle$ QED

(b) $\hat{N}\hat{a}^\dagger|\beta\rangle \overset{(180)}{=} \hat{a}^\dagger\hat{a}\hat{a}^\dagger|\beta\rangle$

From (178), $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

$= \hat{a}^\dagger(\hat{a}^\dagger\hat{a} + 1)|\beta\rangle$

$= \hat{a}^\dagger(\hat{N} + 1)|\beta\rangle$ ... now use (182)

$= \hat{a}^\dagger(\beta + 1)|\beta\rangle$

$= (\beta + 1)\hat{a}^\dagger|\beta\rangle$ QED

Exercise #52

(a) From (185a), namely $\hat{N}\hat{a}|\beta\rangle = (\beta-1)\hat{a}|\beta\rangle$, we see that $\hat{a}|\beta\rangle$ is an eigenstate of the number operator with one less quantum of excitation than $|\beta\rangle$ (see (182) regarding this latter point). Hence we may write

(a) $\hat{a}|\beta\rangle = \gamma|\beta-1\rangle$, where γ is a constant that is to be determined. Form the Hermitian conjugate of (a), namely:

(b) $\langle\beta|\hat{a}^\dagger = \langle\beta-1|\gamma^*$, and then "bra-ket" (b) with (a), so that:

(c) $\langle\beta|\hat{a}^\dagger\hat{a}|\beta\rangle = \langle\beta-1|\gamma^*\gamma|\beta-1\rangle$

$\langle\beta|\hat{N}|\beta\rangle = |\gamma|^2\underbrace{\langle\beta-1|\beta-1\rangle}$ (180)

$\langle\beta|\beta|\beta\rangle = |\gamma|^2$ (182)

γ chosen such that this is unity, i.e. $|\beta\rangle, |\beta-1\rangle$ etc. are all normalized

$\beta\langle\beta|\beta\rangle = |\gamma|^2$

$|\gamma|^2 = \beta$

$\gamma = \sqrt{\beta}$ → choose γ to be real and non-negative

(a) $\Rightarrow \hat{a}|\beta\rangle = \sqrt{\beta}|\beta-1\rangle$ QED

(b) logic similar to (a)! Eq. (185b) tells us that $\hat{a}^\dagger|\beta\rangle$ is an eigenfunction of $\hat{N}$, with one more quantum of excitation than $|\beta\rangle$ (cf 182). Hence we may write:
(d) $\hat{a}^\dagger|\beta\rangle = \eta|\beta+1\rangle$, where η is to be determined. The Hermitian conjugate of (d) is: (e) $\langle\beta|\hat{a} = \eta^*\langle\beta+1|$;
"bra-ket" (e) with (d) to get: (f) $\langle\beta|\hat{a}\;\hat{a}^\dagger|\beta\rangle = |\eta|^2\langle\beta+1|\beta+1\rangle$.
Note from (178) that $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}+1 \overset{180}{=} \hat{N}+1$, so that (f) becomes: (g)

$$\langle\beta|\hat{N}+1|\beta\rangle = |\eta|^2\underbrace{\langle\beta+1|\beta+1\rangle}_{1}$$

$$\langle\beta|\beta+1|\beta\rangle = |\eta|^2$$

$$(\beta+1)\underbrace{\langle\beta|\beta\rangle}_{1} = |\eta|^2$$

$$\Rightarrow \eta = \sqrt{\beta+1}$$

again, we demand η be real and non-negative

$\Rightarrow$ from (d), $\hat{a}^\dagger|\beta\rangle = \sqrt{\beta+1}|\beta+1\rangle$, as required.

(c) • We demand $|n\rangle$ be normalised, in (192) — i.e., $\langle n|n\rangle = 1$.

• $\langle n-1|n-1\rangle \overset{193a}{=} \langle n|\frac{1}{\sqrt{n}}\hat{a}^\dagger\frac{1}{\sqrt{n}}\hat{a}|n\rangle = \frac{1}{n}\langle n|\hat{a}^\dagger\hat{a}|n\rangle$
$\overset{180}{=} \frac{1}{n}\langle n|\hat{N}|n\rangle \overset{182}{=} \frac{1}{n}\langle n|n|n\rangle = \frac{n}{n}\langle n|n\rangle \overset{192}{=} 1$. QED!

• $\langle n+1|n+1\rangle \overset{193b}{=} \langle n|\frac{\hat{a}}{\sqrt{n+1}}\frac{\hat{a}^\dagger}{\sqrt{n+1}}|n\rangle = \frac{1}{n+1}\langle n|\hat{a}\hat{a}^\dagger|n\rangle$
$\overset{178}{=} \frac{1}{n+1}\langle n|\hat{a}^\dagger\hat{a}+1|n\rangle \overset{180}{=} \frac{1}{n+1}\langle n|\hat{N}+1|n\rangle \overset{182}{=} \frac{1}{n+1}\langle n|n+1|n\rangle = \frac{n+1}{n+1}\langle n|n\rangle = 1$.
QED!

Exercise #53 Start with the vacuum state $|0\rangle$... hence:

• $\hat{a}^\dagger|0\rangle \overset{193b}{=} \sqrt{0+1}|0+1\rangle = |1\rangle$ (a)

• $(\hat{a}^\dagger)^2|0\rangle = \hat{a}^\dagger(\hat{a}^\dagger|0\rangle) \overset{(a)}{=} \hat{a}^\dagger|1\rangle \overset{193b}{=} \sqrt{1+1}|1+1\rangle = \sqrt{2}|2\rangle$ (b)

• $(\hat{a}^\dagger)^3|0\rangle = \hat{a}^\dagger(\hat{a}^\dagger)^2|0\rangle \overset{(b)}{=} \hat{a}^\dagger\sqrt{2}|2\rangle \overset{193b}{=} \sqrt{2}\sqrt{2+1}|2+1\rangle = \sqrt{3!}|3\rangle$ (c)

• $(\hat{a}^\dagger)^4|0\rangle = \hat{a}^\dagger(\hat{a}^\dagger)^3|0\rangle \overset{(c)}{=} \hat{a}^\dagger\sqrt{3!}|3\rangle \overset{193b}{=} \sqrt{3+1}\sqrt{3!}|3+1\rangle = \sqrt{4!}|4\rangle$ (d)

$\vdots$

Generalisation: $(\hat{a}^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$

$\Rightarrow |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$, as required.

Exercise #54

same as "$\partial/\partial z$" — same as "$\partial/\partial y$"

$$[L_x, L_y] \overset{201a,b}{=\!=} [-i\hbar(y\partial_z - z\partial_y), -i\hbar(z\partial_x - x\partial_z)]$$

$$= -\hbar^2[y\partial_z - z\partial_y, z\partial_x - x\partial_z]$$

$$\Rightarrow \frac{-1}{\hbar^2}[L_x, L_y] = [y\partial_z - z\partial_y, z\partial_x - x\partial_z]$$

... we now freely use the results of exercise #19

$$= [y\partial_z, z\partial_x] - [y\partial_z, x\partial_z] - [z\partial_y, z\partial_x] + [z\partial_y, x\partial_z]$$

$$= [y\partial_z, z]\partial_x + z\cancel{[y\partial_z, \partial_x]}^{0} - \cancel{[y\partial_z, x]}^{0}\partial_z - x[y\partial_z, \partial_z]$$
$$- [z\partial_y, z]\partial_x - z\cancel{[z\partial_y, \partial_x]}^{0} + \cancel{[z\partial_y, x]}^{0}\partial_z + x[z\partial_y, \partial_z]$$

(Note – 4 commutators crossed out, since each contain 3 different variables/operators, all of which commute.)

$$= -[z, y\partial_z]\partial_x + x[\partial_z, y\partial_z] + [z, z\partial_y]\partial_x - x[\partial_z, z\partial_y]$$

$$= -\cancel{[z, y]}^{0}\partial_z\partial_x - y[z, \partial_z]\partial_x + x\cancel{[\partial_z, y]}^{0}\partial_z + xy\cancel{[\partial_z, \partial_z]}^{0}$$
$$+ \cancel{[z, z]}^{0}\partial_y\partial_x + z\cancel{[z, \partial_y]}^{0}\partial_x - x[\partial_z, z]\partial_y - xz\cancel{[\partial_z, \partial_y]}^{0}$$

$$= y[\partial_z, z]\partial_x - x[\partial_z, z]\partial_y \quad (a)$$

Now, $[\partial_z, z]f$ — f: any differentiable function of z

$$= (\partial_z z - z\partial_z)f$$
$$= \partial_z(zf) - z\partial_z f$$
$$= \frac{\partial}{\partial z}(zf) - z\frac{\partial f}{\partial z}$$

use product rule

$$= \cancel{z\frac{\partial f}{\partial z}} + f - \cancel{z\frac{\partial f}{\partial z}}$$
$$= f$$

Thus: $[\partial_z, z]f = 1 \cdot f$

Thus: $[\partial_z, z] = 1$

Hence (a) becomes:

$$\frac{-1}{\hbar^2}[L_x, L_y] = y\partial_x - x\partial_y$$

$$\frac{1}{\hbar^2}[L_x, L_y] = x\partial_y - y\partial_x$$

Now use (201c):

$$\Rightarrow \frac{1}{\hbar^2}[L_x, L_y] = \frac{L_z}{-i\hbar}$$

$$\Rightarrow [L_x, L_y] = \frac{\hbar^2}{-i\hbar}L_z = i\hbar L_z.$$

QED!

Note: Proof of (202b) and (202c) follows from cyclic permutation of x/y/z in the above derivation.

Exercise #55

$$[\vec{L}^2, L_x] \overset{203}{=} [L_x^2 + L_y^2 + L_z^2, L_x]$$
$$= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$$
$$\overset{70a}{=} -[L_x, L_x^2] - [L_x, L_y^2] - [L_x, L_z^2]$$
$$\overset{70e}{=} -\cancel{[L_x, L_x]}L_x - L_x\cancel{[L_x, L_x]}$$
$$- L_y[L_x, L_y] - [L_x, L_y]L_y$$
$$- L_z[L_x, L_z] - [L_x, L_z]L_z \longrightarrow \text{... now use (202a) and (202c) ...}$$
$$= -L_y(i\hbar L_z) - (i\hbar L_z)L_y$$
$$- L_z(-i\hbar L_y) - (-i\hbar L_y)L_z$$

note!

$$= i\hbar(\cancel{-L_yL_z} - \cancel{L_zL_y} + \cancel{L_zL_y} + \cancel{L_yL_z})$$
$$= 0 \quad \text{as required}$$

Analogous proofs hold for the statements $[\vec{L}^2, L_y] = 0$ and $[\vec{L}^2, L_z] = 0$.

Exercise #56

(a) $x = r\sin\theta\cos\phi$

(b) $y = r\sin\theta\sin\phi$

(c) $z = r\cos\theta$

(d) $r = \sqrt{x^2+y^2+z^2}$

(e) $\phi = \tan^{-1}(y/x)$

(f) $\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

Note — (d) through (f) can be written on inspection, from FIG 18.

We now calculate some "intermediate results":

(g) $\frac{\partial r}{\partial x} \overset{(d)}{=} \frac{1}{2} \times \frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \overset{(a)}{=} \frac{r\sin\theta\cos\phi}{r} = \sin\theta\cos\phi$

(h) $\frac{\partial r}{\partial y} \overset{(d)}{=} \frac{y}{r} \overset{(b)}{=} \frac{r\sin\theta\sin\phi}{r} = \sin\theta\sin\phi$

(i) $\frac{\partial r}{\partial z} \overset{(d)}{=} \frac{z}{r} \overset{(c)}{=} \frac{r\cos\theta}{r} = \cos\theta$

(j) $\frac{\partial \phi}{\partial x} \overset{(e)}{=} \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\frac{y}{x} = \frac{-yx^{-2}}{1+(y/x)^2} = \frac{-y}{x^2+y^2} \overset{(a),(b)}{=} \frac{-\sin\phi}{r\sin\theta}$

(k) $\frac{\partial \phi}{\partial y} \overset{(e)}{=} \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\frac{y}{x} = \frac{x^{-1}}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} \overset{(a)(b)}{=} \frac{\cos\phi}{r\sin\theta}$

(l) $\frac{\partial \phi}{\partial z} \overset{(e)}{=} \frac{\partial}{\partial z}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$

$$\text{(m)}\ \frac{\partial\theta}{\partial x} \overset{\text{(f)}}{=} \frac{\partial}{\partial x}\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial x}\frac{\sqrt{x^2+y^2}}{z} = \cdots = \frac{\cos\theta\cos\phi}{r}$$

$$\text{(n)}\ \frac{\partial\theta}{\partial y} \overset{\text{(f)}}{=} \frac{\partial}{\partial y}\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial y}\frac{\sqrt{x^2+y^2}}{z} = \cdots = \frac{\cos\theta\sin\phi}{r}$$

$$\text{(o)}\ \frac{\partial\theta}{\partial z} \overset{\text{(f)}}{=} \frac{\partial}{\partial z}\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \underbrace{\frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial z}}\frac{\sqrt{x^2+y^2}}{z} = \cdots = -\frac{\sin\theta}{r}$$

Now we can start answering the question:

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$= -i\hbar\left\{\underbrace{r\sin\theta\sin\phi}_{\text{(b)}}\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right) - \underbrace{r\cos\theta}_{\text{(c)}}\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right)\right\}$$

now use (i) now use (o) now use (l) now use (h) now use (n) now use (k)

$$= -i\hbar\left\{r\sin\theta\sin\phi\left(\cos\theta\frac{\partial}{\partial r} + -\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + 0\right) - r\cos\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right\}$$

$$= -i\hbar\left\{\cancel{r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r}} - \sin^2\theta\sin\phi\frac{\partial}{\partial\theta} - \cancel{r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r}} - \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right\}$$

$$= -i\hbar\left\{-\sin\phi\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right\}$$

$$= -i\hbar\left\{-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right\}$$ QED! ☺

$$L_y = -i\hbar\left\{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}$$

$$= -i\hbar\left\{\underbrace{r\cos\theta}_{\text{(c)}}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right) - \underbrace{r\sin\theta\cos\phi}_{\text{(a)}}\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}\right)\right\}$$

now use (g) now use (m) now use (j) now use (i) now use (o) now use (l)

$$= -i\hbar\left\{r\cos\theta\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - r\sin\theta\cos\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + 0\right)\right\}$$

$$= -i\hbar\left\{\cancel{r\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial r}} + \cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - \cancel{r\sin\theta\cos\phi\cos\theta\frac{\partial}{\partial r}} + \sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right\}$$

$$= -i\hbar\left\{\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right\}$$

$$= -i\hbar\left\{\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right\}$$ QED! ☺

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -i\hbar\left\{\underbrace{r\sin\theta\cos\phi}_{a}\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\right) - \underbrace{r\sin\theta\sin\phi}_{b}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}\right)\right\}$$

$$= -i\hbar\left\{r\sin\theta\cos\phi\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right.$$
$$\left. - r\sin\theta\sin\phi\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right\}$$

$$= -i\hbar\left\{\cancel{r\sin^2\theta\cos\phi\sin\phi\frac{\partial}{\partial r}} + \cancel{\sin\theta\cos\phi\cos\theta\sin\phi\frac{\partial}{\partial\theta}} + \cos^2\phi\frac{\partial}{\partial\phi}\right.$$
$$\left. - \cancel{r\sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r}} - \cancel{\sin\theta\sin\phi\cos\theta\cos\phi\frac{\partial}{\partial\theta}} + \sin^2\phi\frac{\partial}{\partial\phi}\right\}$$

$$= -i\hbar\underbrace{(\cos^2\phi + \sin^2\phi)}_{1}\frac{\partial}{\partial\phi} = -i\hbar\frac{\partial}{\partial\phi} \quad \text{QED}$$

Consider a well-behaved but otherwise arbitrary function $f = f(\theta, \phi)$.

$$\frac{1}{(-i\hbar)^2}L^2 f$$

$$= \frac{1}{(-i\hbar)^2}(L_x^2 + L_y^2 + L_z^2)f$$

$$= \left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\left(-\sin\phi\frac{\partial f}{\partial\theta} - \cot\theta\cos\phi\frac{\partial f}{\partial\phi}\right)$$
$$+ \left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial f}{\partial\theta} - \cot\theta\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{\partial}{\partial\phi}\frac{\partial f}{\partial\phi}$$

$$= \sin\phi\frac{\partial}{\partial\theta}\left(\sin\phi\frac{\partial f}{\partial\theta}\right) + \sin\phi\frac{\partial}{\partial\theta}\left(\cot\theta\cos\phi\frac{\partial f}{\partial\phi}\right) + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\theta}\right)$$
$$+ \cot\theta\cos\phi\frac{\partial}{\partial\phi}\left(\cot\theta\cos\phi\frac{\partial f}{\partial\phi}\right) + \cos\phi\frac{\partial}{\partial\theta}\left(\cos\phi\frac{\partial f}{\partial\theta}\right) - \cos\phi\frac{\partial}{\partial\theta}\left(\cot\theta\sin\phi\frac{\partial f}{\partial\phi}\right)$$
$$- \cot\theta\sin\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial f}{\partial\theta}\right) + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\left(\cot\theta\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{\partial^2 f}{\partial\phi^2}$$

$$= \sin^2\phi\frac{\partial^2 f}{\partial\theta^2} + \cancel{\sin\phi\cos\phi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial f}{\partial\phi}\right)} + \cot\theta\cos\phi\left(\cos\phi\frac{\partial f}{\partial\theta} + \cancel{\sin\phi\frac{\partial^2 f}{\partial\theta\partial\phi}}\right)$$
$$+ \cot^2\theta\cos\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial f}{\partial\phi}\right) + \cos^2\phi\frac{\partial^2 f}{\partial\theta^2} - \cancel{\cos\phi\sin\phi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial f}{\partial\phi}\right)}$$
$$- \cot\theta\sin\phi\left(-\sin\phi\frac{\partial f}{\partial\theta} + \cancel{\cos\phi\frac{\partial^2 f}{\partial\theta\partial\phi}}\right) + \cot^2\theta\sin\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{\partial^2 f}{\partial\phi^2}$$

$$= (\sin^2\phi + \cos^2\phi)\frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta}(\cos^2\phi + \sin^2\phi)$$
$$+ \cot^2\theta\left[\cos\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial f}{\partial\phi}\right) + \sin\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right)\right] + \frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\left[\cos\phi\left(-\sin\phi\frac{\partial f}{\partial\phi} + \cos\phi\frac{\partial^2 f}{\partial\phi^2}\right) + \sin\phi\left(\cos\phi\frac{\partial f}{\partial\phi} + \sin\phi\frac{\partial^2 f}{\partial\phi^2}\right)\right] + \frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\left[\cancel{-\cos\phi\sin\phi\frac{\partial f}{\partial\phi}} + \cos^2\phi\frac{\partial^2 f}{\partial\phi^2} + \cancel{\sin\phi\cos\phi\frac{\partial f}{\partial\phi}} + \sin^2\phi\frac{\partial^2 f}{\partial\phi^2}\right] + \frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\left[\underbrace{\cos^2\phi + \sin^2\phi}_{1}\right]\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + (1+\cot^2\theta)\frac{\partial^2 f}{\partial\phi^2}$$

$$\left(\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow 1 + \cot^2\theta = \frac{1}{\sin^2\theta}\right)$$

$$= \frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{1}{\sin\theta}\left(\sin\theta\frac{\partial^2 f}{\partial\theta^2} + \sin\theta\cot\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{1}{\sin\theta}\left(\sin\theta\frac{\partial^2 f}{\partial\theta^2} + \frac{\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}}\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{1}{\sin\theta}\left(\sin\theta\frac{\partial^2 f}{\partial\theta^2} + \cos\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Hence: $\vec{L}^2 f = (-i\hbar)^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right\}$

$$= \underbrace{-\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}}_{\text{this is } \vec{L}^2}f$$

this is $\vec{L}^2$, hence:

$$\vec{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \quad \text{QED!}$$

Exercise #57 $[\vec{J}^2, J_x] = [J_x^2 + J_y^2 + J_z^2, J_x]$

$= \cancel{[J_x^2, J_x]} + [J_y^2, J_x] + [J_z^2, J_x]$ ... now use (70a)

$= -[J_x, J_y^2] - [J_x, J_z^2]$ ... now use (70c)

$= -[J_x, J_y]J_y - J_y[J_x, J_y]$ ... now use (213a)

$\quad - [J_x, J_z]J_z - J_z[J_x, J_z]$ ... now use (213c)

$= -(i\hbar J_z)J_y - J_y(i\hbar J_z) - (-i\hbar J_y)J_z - J_z(-i\hbar J_y)$

$= i\hbar\{-\cancel{J_z J_y} - \cancel{J_y J_z} + \cancel{J_y J_z} + \cancel{J_z J_y}\}$

$= 0$, as required.

Analogous method of proof for remaining equations in (215).

Exercise #58

$$[\vec{J}^2, J_\pm] \overset{219}{=} [\vec{J}^2, J_x \pm iJ_y] = \overbrace{[\vec{J}^2, J_x]}^{\text{zero, by 215}} \pm i\overbrace{[\vec{J}^2, J_y]}^{} = 0 \quad \text{QED}$$

$$J_\pm J_\mp \overset{219}{=} (J_x \pm iJ_y)(J_x \mp iJ_y) + \underbrace{J_z^2 - J_z^2}$$

this is zero, so we can "add it on the end!"

$$= J_x^2 \mp iJ_xJ_y \pm iJ_yJ_x - i^2J_y^2 + J_z^2 - J_z^2$$

$$= \underline{J_x^2 + J_y^2 + J_z^2} \mp i(J_xJ_y - J_yJ_x) - J_z^2$$

$$= \vec{J}^2 \mp i[J_x, J_y] - J_z^2 \quad \text{... now use (213a)}$$

$$= \vec{J}^2 \mp i(i\hbar J_z) - J_z^2 = \vec{J}^2 \pm \hbar J_z - J_z^2 \quad \text{QED}$$

$$[J_z, J_\pm] \overset{219}{=} [J_z, J_x \pm iJ_y]$$

$$= [J_z, J_x] \pm i[J_z, J_y] \quad \text{... now use (213b) \& (213c)}$$

$$= i\hbar J_y \pm i(-i\hbar J_x)$$

$$= \hbar(\pm J_x + iJ_y) = \pm\hbar(J_x \pm iJ_y) \overset{219}{=} \pm\hbar J_\pm \quad \text{QED}$$

Exercise #59

"J_- annihilates $|jm_{min}\rangle$" can be written as:

$J_-|jm_{min}\rangle = 0$. Now apply J_+ to both sides, hence

$J_+J_-|jm_{min}\rangle = 0$ (cf 230) so that:

$$0 = J_+J_-|jm_{min}\rangle \quad \text{... now use (220b)}$$

$$= \{\vec{J}^2 - J_z^2 + \hbar J_z\}|jm_{min}\rangle \quad \text{... now use (216a, b)}$$

$$= \underline{\{j(j+1) - m_{min}^2 + m_{min}\}}\hbar^2|jm_{min}\rangle.$$

This quantity in braces must vanish, yielding the following quadratic in m_{min}:

$-m_{min}^2 + m_{min} + j(j+1) = 0$. Hence, from the quadratic formula,

$$m_{min} = \frac{-1 \pm \sqrt{1 - 4(-1)j(j+1)}}{-2} = \frac{1}{2} \mp \frac{1}{2}\sqrt{4j^2 + 4j + 1} = \frac{1}{2} \mp \frac{1}{2}\sqrt{(2j+1)^2}$$

$= \frac{1}{2} \mp \frac{1}{2}(2j+1) = -j$ or $j+1$. Rejecting the solution "$j+1$", because it is larger than the largest m-value (j), we have $m_{min} = -j$, as required.

Exercise #60

Combining (240) and (246), we have:

$$J_\pm|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\,|j, m\pm1\rangle \quad (a).$$

(a) As a first special case of (a), let $m = m_{max} = j$, and take the "+" sign of the "+". Thus $J_+|j\,m_{max}\rangle = \hbar\sqrt{j(j+1) - m_{max}(m_{max}+1)}\,|j, m_{max}+1\rangle$

$$= \hbar\sqrt{j(j+1) - j(j+1)}\,|j, \underbrace{m_{max}}_{j}+1\rangle$$

$= 0$, as required.

(b) As a second special case of ⓐ, let $m = m_{min} = -j$, and take the "−" sign of the "±". Thus:

$$J_-|j\,m_{min}\rangle = \hbar\sqrt{j(j+1) - m_{min}(m_{min}-1)}\,|j, m_{min}-1\rangle$$

$$= \hbar\sqrt{j(j+1) - -j(-j-1)}\,|j, \underbrace{m_{min}}_{-j}-1\rangle$$

$$= \hbar\sqrt{j(j+1) - j(j+1)}\,|j, m_{min}-1\rangle$$

$= 0$, as required.

Exercise #61 Equation 268 when $s=1$, equation (261) becomes:

$$\langle \chi_{\underset{1}{s'}, m_s'} | \vec{S}^2 | \chi_{\underset{1}{s}, m_s}\rangle = s(s+1)\hbar^2\delta_{ss'}\delta_{m_s m_s'} = 1(1+1)\hbar^2\delta_{ss'}\delta_{m_s m_s'}$$

$$= 2\hbar^2\delta_{ss'}\delta_{m_s m_s'}$$

$$= 2\hbar^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Equation 269 when $s=1$, equation (262) becomes:

$\langle \chi_{\underset{1}{s'}, m_s'} | S_z | \chi_{\underset{1}{s}, m_s}\rangle = m_s\hbar\,\delta_{ss'}\delta_{m_s m_s'}$. Now, the presence of the factor $\delta_{ss'}\delta_{m_s m_s'}$, in the above equation, tells us that the (3×3) matrix for S_z is diagonal. The diagonal values are $m_s\hbar$. Since $m_s = -1, 0, 1$ (see 266), the required matrix is $\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$, as required.

Equation 270 when $s=1$, equation (264) becomes:

ⓐ $$\langle \chi_{s'=1, m_s'} | S_x | \chi_{s=1, m_s}\rangle = \frac{1}{2}\hbar\sqrt{2 - m_s(m_s+1)}\,\delta_{m_s+1, m_s'} + \frac{1}{2}\hbar\sqrt{2 - m_s(m_s-1)}\,\delta_{m_s-1, m_s'}$$

$$= \frac{\hbar}{2}\left(\sqrt{2 - m_s(m_s+1)}\,\delta_{m_s+1, m_s'} + \sqrt{2 - m_s(m_s-1)}\,\delta_{m_s-1, m_s'}\right).$$

Now, the Kronecker deltas — in the above formula — tell us that that the only non-zero matrix elements lie only in those "slots" that are coloured in here: $\begin{pmatrix}\circ&\bullet&\circ\\ \bullet&\circ&\bullet\\ \circ&\bullet&\circ\end{pmatrix}$. More explicitly, the required matrix is: ↪ cf (174)

ⓑ

$$S_x = \begin{pmatrix} 0 & \langle \chi_{s'=1, m_s'=1} | S_x | \chi_{s=1, m_s=0}\rangle & 0 \\ \langle \chi_{s'=1, m_s'=0} | S_x | \chi_{s=1, m_s=1}\rangle & 0 & \langle \chi_{s'=1, m_s'=0} | S_x | \chi_{s=1, m_s=-1}\rangle \\ 0 & \langle \chi_{s'=1, m_s'=-1} | S_x | \chi_{s=1, m_s=0}\rangle & 0 \end{pmatrix}$$

Now use ⓐ. Hence:

$$S_x = \left(\begin{array}{c|c|c} 0 & \frac{\hbar}{2}\left(\begin{array}{l}\sqrt{2-0(0+1)}\,\delta_{0+1,1}\\ +\sqrt{2-0(0-1)}\,\delta_{0-1,1}\end{array}\right) & 0 \\ \hline \frac{\hbar}{2}\left(\begin{array}{l}\sqrt{2-1(1+1)}\,\delta_{1+1,0}\\ +\sqrt{2-1(1-1)}\,\delta_{1-1,0}\end{array}\right) & 0 & \frac{\hbar}{2}\left(\begin{array}{l}\sqrt{2--1(-1+1)}\,\delta_{-1+1,0}\\ +\sqrt{2--1(-1-1)}\,\delta_{-1-1,0}\end{array}\right) \\ \hline 0 & \frac{\hbar}{2}\left(\begin{array}{l}\sqrt{2-0(0+1)}\,\delta_{0+1,-1}\\ +\sqrt{2-0(0-1)}\,\delta_{0-1,-1}\end{array}\right) & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} 0 & \frac{\hbar}{2}\sqrt{2} & 0 \\ \hline \frac{\hbar}{2}\sqrt{2} & 0 & \frac{\hbar}{2}\sqrt{2} \\ \hline 0 & \frac{\hbar}{2}\sqrt{2} & 0 \end{array}\right) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ as required.}$$

Equation 271 When $s=1$, equation (265) becomes:

$$ⓒ \quad \langle \chi_{1,m_s'} | S_y | \chi_{1,m_s} \rangle = \frac{\hbar}{2i}\left[\sqrt{2-m_s(m_s+1)}\,\delta_{m_s+1,m_s'} - \sqrt{2-m_s(m_s-1)}\,\delta_{m_s-1,m_s'}\right].$$

Again, the Kronecker deltas in ⓒ tell us immediately that we need only seek non zero matrix elements (of S_y) in the coloured-in "slots": $\begin{pmatrix} \circ & \bullet & \circ \\ \bullet & \circ & \bullet \\ \circ & \bullet & \circ \end{pmatrix}$. From ⓒ, the required matrix is:

$$S_y = \left(\begin{array}{c|c|c} 0 & \langle \chi_{1,1} | S_y | \chi_{1,0} \rangle & 0 \\ \hline \langle \chi_{1,0} | S_y | \chi_{1,1} \rangle & 0 & \langle \chi_{1,0} | S_y | \chi_{1,-1} \rangle \\ \hline 0 & \langle \chi_{1,-1} | S_y | \chi_{1,0} \rangle & 0 \end{array}\right)$$

... now use ⓒ

$$= \left(\begin{array}{c|c|c} 0 & \frac{\hbar}{2i}\left(\begin{array}{l}\sqrt{2-0(0+1)}\,\delta_{0+1,1}\\ -\sqrt{2-0(0-1)}\,\delta_{0-1,1}\end{array}\right) & 0 \\ \hline \frac{\hbar}{2i}\left(\begin{array}{l}\sqrt{2-1(1+1)}\,\delta_{1+1,0}\\ -\sqrt{2-1(1-1)}\,\delta_{1-1,0}\end{array}\right) & 0 & \frac{\hbar}{2i}\left(\begin{array}{l}\sqrt{2--1(-1+1)}\,\delta_{-1+1,0}\\ -\sqrt{2--1(-1-1)}\,\delta_{-1-1,0}\end{array}\right) \\ \hline 0 & \frac{\hbar}{2i}\left(\begin{array}{l}\sqrt{2-0(0+1)}\,\delta_{0+1,-1}\\ -\sqrt{2-0(0-1)}\,\delta_{0-1,-1}\end{array}\right) & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} 0 & \frac{\hbar}{2i}\sqrt{2} & 0 \\ \hline \frac{\hbar}{2i}(-\sqrt{2}) & 0 & \frac{\hbar}{2i}\sqrt{2} \\ \hline 0 & \frac{\hbar}{2i}(-\sqrt{2}) & 0 \end{array}\right) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \text{ as required.}$$

Equation 253a

$$[S_x, S_y] = S_x S_y - S_y S_x \overset{\substack{270\\271}}{=} \frac{\hbar^2}{2}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

$$=\frac{\hbar^2}{2}\left[\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}\right] = \frac{\hbar^2}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix}$$

$$= i\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overset{269}{=} i\hbar S_z, \text{ as required.}$$

Equations 253 b, c similar to demonstration of 253a!

Equation 254 $S_x^2 + S_y^2 + S_z^2$ ... now use (269) | (270) | (271)

$$= \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \overset{268}{=} \vec{S}^2$$

Equation 255

$$[\vec{S}^2, S_x] = \vec{S}^2 S_x - S_x \vec{S}^2 \quad \text{... now use (268) and (270)}$$

$$= \frac{2\hbar^3}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \frac{2\hbar^3}{\sqrt{2}}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right] = 0,$$

as required. Analogous proofs for "$[\vec{S}^2, S_y] = 0$" and "$[\vec{S}^2, S_z] = 0$".

Equation 256a when $s=1$, this equation becomes: (d) $\vec{S}^2 \chi_{1,m_s} = s(s+1)\hbar^2 \chi_{1,m_s}$

There are three cases of this equation, corresponding to $m_s = 1, 0, -1$.
We separately verify each case, below.

* When $m_s=1$, $\vec{S}^2\chi_{1,1} \overset{267/8}{=} 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1(1+1)\hbar^2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \overset{1}{s}(\overset{1}{s}+1)\hbar^2\chi_{1,1}$ ✓
* When $m_s=0$, $\vec{S}^2\chi_{1,0} \overset{267/8}{=} 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1(1+1)\hbar^2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \overset{1}{s}(\overset{1}{s}+1)\hbar^2\chi_{1,0}$ ✓
* When $m_s=-1$, $\vec{S}^2\chi_{1,-1} \overset{267/8}{=} 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1(1+1)\hbar^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \overset{1}{s}(\overset{1}{s}+1)\hbar^2\chi_{1,-1}$ ✓

Equation 256b when $s=1$, this equation becomes: (e) $S_z\chi_{1,m_s} = m_s\hbar\chi_{1,m_s}$.

Below, we separately consider the three cases ($m_s = 1, 0, -1$) of (e):

* When $m_s=1$, $S_z\chi_{1,1} \overset{267/9}{=} \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1)\hbar\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = m_s^{=1}\hbar\chi_{1,1}$ ✓
* When $m_s=0$, $S_z\chi_{1,0} \overset{267/9}{=} \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0)\hbar\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = m_s^{=0}\hbar\chi_{1,0}$ ✓
* When $m_s=-1$, $S_z\chi_{1,-1} \overset{267/9}{=} \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (-1)\hbar\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = m_s^{=-1}\hbar\chi_{1,-1}$ ✓

Exercise #62 Assuming the wavefunction to be normalised to unity, we interpret $|\Psi_{m_s}(\vec{r},t)|^2$ as the probability of finding the particle at time t, in the volume element $d\vec{r}$, with the component of its spin along the z axis equal to $m_s\hbar$, where $m_s = -1, 0$ or $+1$.

Exercise #63 (a) Equation 276 From the $s=\frac{1}{2}$ case of equation (264), we have:

$$\langle \chi_{s'=\frac{1}{2},m_s'=\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=\frac{1}{2}} \rangle \overset{264}{=} 0$$
$$\langle \chi_{s'=\frac{1}{2},m_s'=-\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=-\frac{1}{2}} \rangle \overset{264}{=} 0$$
"Diagonal" elements vanish

$$\langle \chi_{s'=\frac{1}{2},m_s'=\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=-\frac{1}{2}} \rangle \overset{264}{=} \frac{1}{2}\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)--\frac{1}{2}\left(-\frac{1}{2}+1\right)}\,\overbrace{\delta_{\frac{1}{2},\frac{1}{2}}\delta_{-\frac{1}{2}+1,\frac{1}{2}}}^{1}$$
$$+\frac{1}{2}\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)--\frac{1}{2}\left(-\frac{1}{2}-1\right)}\,\underbrace{\delta_{\frac{1}{2},\frac{1}{2}}\delta_{-\frac{1}{2}-1,\frac{1}{2}}}_{0}$$
$$=\frac{1}{2}\hbar\sqrt{\frac{1}{2}\left(\frac{3}{2}\right)+\frac{1}{2}\left(\frac{1}{2}\right)} = \frac{1}{2}\hbar$$

$$\langle \chi_{s'=\frac{1}{2},m_s'=-\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=\frac{1}{2}} \rangle \overset{264}{=} \frac{1}{2}\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}+1\right)}\,\overbrace{\delta_{\frac{1}{2},\frac{1}{2}}\delta_{\frac{1}{2}+1,\frac{1}{2}}}^{0}$$
$$+\frac{1}{2}\hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\,\underbrace{\delta_{\frac{1}{2},\frac{1}{2}}\delta_{\frac{1}{2}-1,-\frac{1}{2}}}_{1}$$
$$=\frac{1}{2}\hbar\sqrt{\frac{1}{2}\left(\frac{3}{2}\right)--\frac{1}{2}} = \frac{1}{2}\hbar$$

Hence the matrix representing S_x, is:

$$S_x = \begin{pmatrix} \langle \chi_{s'=\frac{1}{2},m_s'=\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=\frac{1}{2}} \rangle & \langle \chi_{s'=\frac{1}{2},m_s'=\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=-\frac{1}{2}} \rangle \\ \langle \chi_{s'=\frac{1}{2},m_s'=-\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=\frac{1}{2}} \rangle & \langle \chi_{s'=\frac{1}{2},m_s'=-\frac{1}{2}} | S_x | \chi_{s=\frac{1}{2},m_s=-\frac{1}{2}} \rangle \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{1}{2}\hbar \\ \frac{1}{2}\hbar & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{QED}$$

Equations 277–281 Proof similar to that above! The "relevant formulae", respectively, are (265), (262), (261), (263) and (263).

(b) Simple matrix algebra! For example, consider (254): $\vec{S}^2 = S_x^2 + S_y^2 + S_z^2$. This is obeyed by the 2×2 matrices in eqns (276) – (279). Explicitly,

$$\vec{S}^2 \underset{279}{=} \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_x^2 + S_y^2 + S_z^2 = \overbrace{\frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{276} + \overbrace{\frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}^{277} + \overbrace{\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{278}$$
$$= \frac{\hbar^2}{4}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{279}{=} \vec{S}^2 \quad \text{QED.}$$

(c) $S_+ \chi_{down} \overset{274a}{\equiv} S_+ \chi_{\frac{1}{2},-\frac{1}{2}}$

$$\overset{280}{\underset{275}{=}} \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{275}{=} \hbar \chi_{up} \quad \text{QED}$$

$$S_- \chi_{up} \overset{275}{\underset{281}{=}} \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \overset{275}{=} \hbar \chi_{down} \quad \text{QED}$$

(d) $S_+ \chi_{up} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ QED

$S_- \chi_{down} = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ QED

Exercise #64 • $[\sigma_x, \sigma_y] \overset{283}{=} [\frac{2}{\hbar} S_x, \frac{2}{\hbar} S_y] = \frac{4}{\hbar^2}[S_x, S_y] \overset{253a}{=} \frac{4}{\hbar^2}(i\hbar S_z)$

$\overset{283}{=} \frac{4}{\hbar^2} \times i\hbar \times \frac{\hbar}{2} \sigma_z = 2i\sigma_z$

• $[\sigma_y, \sigma_z] = [\frac{2}{\hbar} S_y, \frac{2}{\hbar} S_z] = \frac{4}{\hbar^2}[S_y, S_z] = \frac{4}{\hbar^2}(i\hbar S_x) = \frac{4}{\hbar^2}(i\hbar)(\frac{\hbar}{2}\sigma_x) = 2i\sigma_x$

• $[\sigma_z, \sigma_x] = [\frac{2}{\hbar} S_z, \frac{2}{\hbar} S_x] = \frac{4}{\hbar^2}[S_z, S_x] = \frac{4}{\hbar^2}(i\hbar S_y) = \frac{4}{\hbar^2}(i\hbar)(\frac{\hbar}{2}\sigma_y) = 2i\sigma_y$

• $\sigma_x^2 \overset{283}{=} \left(\frac{2}{\hbar}\right)^2 S_x^2 \overset{276}{=} \left(\frac{2}{\hbar}\right)^2 \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

Note, the number 1, and the 2×2 unit matrix have the same effect when acting on numbers, column vectors (2×1), and 2×2 matrices.

• Similar demonstration that
$\sigma_y^2 = \sigma_z^2 = 1$

• When $i=j$, we have $\{\sigma_i, \sigma_i\} \equiv (\sigma_i)^2 + (\sigma_i)^2$
$= 1 + 1 = 2,$

which is the "$i=j$" special case of (288) ✓.

When $i \neq j$, we need to show that $\{\sigma_i, \sigma_j\}$ vanishes. Do this by enumerating all cases:

① $\{\sigma_x, \sigma_y\} \equiv \sigma_x\sigma_y + \sigma_y\sigma_x = \frac{4}{\hbar^2}(S_x S_y + S_y S_x)$

$= \frac{4}{\hbar^2} \times \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0$

② $\{\sigma_y, \sigma_x\} = \{\sigma_x, \sigma_y\} = 0$, from ①

③ etc... ← similar logic to that given above!

Exercise #65 We need to show that:

① $[H, \vec{L}^2] = 0$ ③ $[\vec{L}^2, L_z] = 0$.

② $[H, L_z] = 0$

① From (293), we may write

$$H = \hat{O}(r) + \frac{\vec{L}^2}{2m_e r^2}, \text{ where } \hat{O}(r) \equiv \frac{-\hbar^2}{2m_e r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r).$$

Note that: $\hat{O}$ depends only on r, and $\vec{L}^2$ depends only on θ and ϕ (see 206d).

Thus: $[H, \vec{L}^2] \equiv H\vec{L}^2 - \vec{L}^2 H$

$$= \left(\hat{O}(r) + \frac{\vec{L}^2(\theta,\phi)}{2m_e r^2}\right)\vec{L}^2 - \vec{L}^2\left(\hat{O}(r) + \frac{\vec{L}^2(\theta,\phi)}{2m_e r^2}\right)$$

$$= \hat{O}\vec{L}^2 + \cancel{\frac{\vec{L}^4}{2m_e r^2}} - \vec{L}^2\hat{O} - \cancel{\frac{\vec{L}^4}{2m_e r^2}}$$

$$= \hat{O}\vec{L}^2 - \vec{L}^2\hat{O}$$

$= 0$ (because $\vec{L}^2$ acts only on θ & ϕ, and $\hat{O}$ " " " r)

② $[H, L_z] \equiv HL_z - L_z H$

$$= \left(\hat{O} + \frac{\vec{L}^2}{2m_e r^2}\right)L_z - L_z\left(\hat{O} + \frac{\vec{L}^2}{2m_e r^2}\right)$$ ... now use (206c)

$$\frac{[H, L_z]}{-i\hbar} = \left(\hat{O} + \frac{\vec{L}^2}{2m_e r^2}\right)\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}\left(\hat{O} + \frac{\vec{L}^2}{2m_e r^2}\right)$$

$$= \underbrace{\hat{O}\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}\hat{O}}_{\text{zero, since } \hat{O} \text{ acts only on } r, \text{ and } \frac{\partial}{\partial\phi} \text{ " " " } \phi.} + \frac{1}{2m_e r^2}\underbrace{\left[\vec{L}^2, \frac{\partial}{\partial\phi}\right]}_{\text{zero, since } [\vec{L}^2, L_z] = 0 \text{ (see 204)}}$$

$= 0$

③ $[\vec{L}^2, L_z] = 0 \rightarrow$ see eq. (204)!

Exercise #66 (a) Equation 294b

$$\vec{J}^2\psi_{E\ell m} \overset{295}{=} \vec{L}^2 R_{E\ell m}Y_{\ell m} = R_{E\ell m}\vec{L}^2 Y_{\ell m} \overset{207}{=} R_{E\ell m}\,\ell(\ell+1)\hbar^2 Y_{\ell m} \overset{295}{=} \ell(\ell+1)\hbar^2\psi_{E\ell m}$$

(b) Equation 294c

$$L_z \psi_{El m} \overset{295}{=} L_z R_{El m} Y_{l m} = R_{El m} L_z Y_{l m} \overset{208}{=} R_{El m}\, m\hbar\, Y_{l m} \overset{295}{=} m\hbar\, \psi_{El m}$$

Exercise #67 Substitute (299) into (298), hence:

$$\left\{ \frac{-\hbar^2}{2m_e r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) + \frac{l(l+1)\hbar^2}{2m_e r^2} + V(r) \right\} \frac{u_{El}(r)}{r} = \frac{E u_{El}(r)}{r}$$

$$\frac{-\hbar^2}{2m_e r^2}\underbrace{\frac{d}{dr}\left(r^2 \frac{d}{dr}\left(\frac{u_{El}}{r}\right)\right)}_{} + \frac{l(l+1)\hbar^2 u_{El}}{2m_e r^3} + \frac{V u_{El}}{r} = \frac{E u_{El}}{r}$$

Now, $\frac{d}{dr}\left(r^2 \frac{d}{dr}\left(\frac{u_{El}}{r}\right)\right) = \frac{d}{dr}\left(r^2\left(\frac{r u_{El}' - u_{El}}{r^2}\right)\right)$ — dash denotes d/dr

$$= \frac{d}{dr}\left(r u_{El}' - u_{El}\right) = r u_{El}'' + \cancel{u_{El}'} - \cancel{u_{El}'} = r u_{El}''$$

$$\frac{-\hbar^2}{2m_e r^{\cancel{2}}}\left(\cancel{r} u_{El}''\right) + \frac{l(l+1)\hbar^2 u_{El}}{2m_e r^{\cancel{3}\,2}} + \frac{V u_{El}}{\cancel{r}} = \frac{E u_{El}}{\cancel{r}}$$

$$\left(\frac{-\hbar^2}{2m_e}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2m_e r^2} + V\right) u_{El} = E u_{El} \qquad \text{QED!}$$

Exercise #68 Begin with (301): (a) $\left(\frac{-\hbar^2}{2m_e}\frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2m_e r^2}\right) u_{El} = E u_{El}$.

As $r \to 0$, suppose that $V(r)$ is no more singular than r^{-1}. Thus $V(r)$ may be neglected in comparison to $l(l+1)\hbar^2/(2m_e r^2)$, as $r \to 0$. Also, in this same limit, E may be neglected. Thus (a) becomes: (b) $\frac{d^2}{dr^2} u_{El} = \frac{l(l+1)}{r^2} u_{El}$, $r \to 0$.

Choosing $u_{El} = \alpha r^{l+1}$, with α some constant, is the solution to (b) which vanishes at $r = 0$ (cf (300)).

Exercise #69 From (306) and (308), we have:

$$0 = \frac{d^2}{d\left(\frac{\hbar\rho}{\sqrt{-8m_eE}}\right)^2} u_{El}(\rho) + \frac{m_e e^2}{2\pi\varepsilon_0 \hbar^2} \times \frac{\sqrt{-8m_eE}}{\hbar\rho} \times u_{El}(\rho) - \frac{l(l+1)}{\hbar^2\rho^2}(-8m_eE) u_{El}(\rho) + \frac{2m_e E u_{El}(\rho)}{\hbar^2}$$

$$0=\left[-\frac{8m_eE}{\hbar^2}\frac{d^2}{d\rho^2}+\frac{m_e e^2\sqrt{-8m_eE}}{2\pi\varepsilon_0\hbar^3\rho}+\frac{8m_eE\,l(l+1)}{\hbar^2\rho^2}+\frac{2m_eE}{\hbar^2}\right]u_{El}(\rho)$$

Now, with an eye on (309), it is evident that we need to "clear away" the factors in front of $d^2/d\rho^2$. To this end, multiply both sides by $-\hbar^2/(8m_eE)$. Thus:

$$0=\left[\frac{d^2}{d\rho^2}-\frac{\cancel{\hbar^2}}{8\cancel{m_e}E}\cdot\frac{\cancel{m_e}e^2\sqrt{-8m_eE}}{2\pi\varepsilon_0\hbar^{\cancel{3}}\rho}-\frac{\cancel{\hbar^2}}{\cancel{8m_eE}}\cdot\frac{\cancel{8m_eE}\,l(l+1)}{\cancel{\hbar^2}\rho^2}-\frac{\cancel{\hbar^2}}{8\cancel{m_eE}}\cdot\frac{2\cancel{m_eE}}{\cancel{\hbar^2}}\right]u_{El}(\rho)$$

Cancelling some factors, we obtain:

$$0=\left[\frac{d^2}{d\rho^2}+\underbrace{\frac{e^2\sqrt{-8m_eE}}{2\times4\times\sqrt{E}\sqrt{E}\times2\pi\varepsilon_0\hbar\rho}}-\frac{l(l+1)}{\rho^2}-\frac14\right]u_{El}(\rho).$$

Nearly there! ☺

Now, this equals: $\dfrac{e^2\sqrt{8}\sqrt{m_e}}{\sqrt{16}\sqrt{-E}\,4\pi\varepsilon_0\hbar\rho}=\dfrac{e^2}{4\pi\varepsilon_0\hbar}\sqrt{\dfrac{m_e}{-2E}}\,\dfrac{1}{\rho}=\dfrac{\lambda}{\rho}$

$$0=\left[\frac{d^2}{d\rho^2}+\frac{\lambda}{\rho}-\frac{l(l+1)}{\rho^2}-\frac14\right]u_{El}(\rho)$$

, as required.

Note: You need to bear in mind the fact that $E<0$! Eg: $-E=\sqrt{E^2}$

Exercise #70 Substitute (311) into (310), to see that (311) indeed obeys (310). Explicitly, we have:

$$\left(\frac{d^2}{d\rho^2}-\frac14\right)e^{\pm\rho/2}=\left[\left(\pm\frac12\right)^2-\frac14\right]e^{\pm\rho/2}=\left[\frac14-\frac14\right]e^{\pm\rho/2}=0 \quad \text{QED}$$

Exercise #71 We begin by focusing on the first term of (313). Bearing in mind the identity $(AB)''=AB''+\overset{BA''}{\cancel{BA}}+2A'B'$ — which can easily be obtained by applying the product rule twice, we see that:

$$\frac{d^2}{d\rho^2}\left[e^{-\rho/2}\sum_{k=0}^{\infty}g_k\rho^{k+l+1}\right]=e^{-\rho/2}\frac{d^2}{d\rho^2}\left[\sum_{k=0}^{\infty}g_k\rho^{k+l+1}\right]+\left[\sum_{k=0}^{\infty}g_k\rho^{k+l+1}\right]\frac{d^2}{d\rho^2}e^{-\rho/2}$$

$$+2\left[\frac{d}{d\rho}e^{-\rho/2}\right]\left[\frac{d}{d\rho}\sum_{k=0}^{\infty}g_k\rho^{k+l+1}\right]$$

$$=e^{-\rho/2}\sum_{k=0}^{\infty}g_k(k+l+1)(k+l)\rho^{k+l-1}+\frac14e^{-\rho/2}\sum_{k=0}^{\infty}g_k\rho^{k+l+1}-e^{-\rho/2}\sum_{k=0}^{\infty}g_k(k+l+1)\rho^{k+l}$$

$$=e^{-\rho/2}\left[\sum_{k=0}^{\infty}g_k(k+l+1)(k+l)\rho^{k+l-1}+\frac14\sum_{k=0}^{\infty}g_k\rho^{k+l+1}-\sum_{k=0}^{\infty}g_k(k+l+1)\rho^{k+l}\right]$$

Put this into (313), and then cancel the exponentials. Thus:

$$\sum_{k=0}^{\infty}g_k(k+l+1)(k+l)\rho^{k+l-1}+\cancel{\frac14\sum_{k=0}^{\infty}g_k\rho^{k+l+1}}-\sum_{k=0}^{\infty}g_k(k+l+1)\rho^{k+l}$$

$$-l(l+1)\sum_{k=0}^{\infty}g_k\rho^{k+l-1}+\lambda\sum_{k=0}^{\infty}g_k\rho^{k+l}-\cancel{\frac14\sum_{k=0}^{\infty}g_k\rho^{k+l+1}}=0$$

Combining the first and third terms, together with the second and fourth, we have:

$$\sum_{k=0}^{\infty} g_k\left[(k+l+1)(k+l)-l(l+1)\right]\rho^{k+l-1} + \sum_{k=0}^{\infty} g_k(\lambda-k-l-1)\rho^{k+l} = 0.$$

Now cancel the ρ^l:

$$\sum_{k=0}^{\infty} g_k\left[(k+l+1)(k+l)-l(l+1)\right]\rho^{k-1} + \sum_{k=0}^{\infty} g_k\left[\lambda-k-l-1\right]\rho^{k} = 0.$$

Now, we want to combine the above two sums, into a single sum. In the first summation, let $k' \equiv k-1$. Thus:

$$\sum_{k'=-1}^{\infty} g_{k'+1}\left[(k'+1+l+1)(k'+1+l)-l(l+1)\right]\rho^{k'} + \sum_{k=0}^{\infty} g_k\left[\lambda-k-l-1\right]\rho^{k} = 0$$

now, when $k'=-1$, the term in square brackets is zero. Hence we may replace the lower limit with $k'=0$.

Drop the primes on the first summation, and replace the lower limit with 0. The two summations can then be combined, thus:

$$\sum_{k=0}^{\infty} \rho^k\left\{g_{k+1}\left[(k+1+l+1)(k+1+l)-l(l+1)\right] + g_k\left[\lambda-k-l-1\right]\right\} = 0$$

this equals $(k+2+l)(k+1+l)-l(l+1)$

$$= k^2+k+kl+2k+2+2l+lk+\cancel{l}+\cancel{l^2}-\cancel{l^2}-\cancel{l}$$

$$= k(k+1)+2kl+2k+2+2l$$

$$= k(k+1)+2l(k+1)+2(k+1)$$

$$= k(k+1)+(k+1)(2l+2)$$

Hence,

$$\sum_{k=0}^{\infty} \rho^k\left\{g_{k+1}\left[k(k+1)+(2l+2)(k+1)\right] + g_k\left[\lambda-k-l-1\right]\right\} = 0,$$ as required.

Exercise #72