PHS3131 – "Relativistic particles and fields"
SOLUTIONS TO EXERCISES

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Exercise #1 Potential difference 7/3/2

= 100 volts

≡ 100 Joules per Coulomb

⇒ an electron of charge "e" picks up 100 e coulombs, which is converted to kinetic energy

$\Rightarrow 100e = \frac{1}{2} m_e v^2$ → electron speed, electron mass

$\Rightarrow v = \sqrt{200e/m_e}$

Now, $e = 1.6 \times 10^{-19}$ coulombs

$m_e = 9.1 \times 10^{-31}$ kg.

$\Rightarrow v = \sqrt{200 \times (1.6 \times 10^{-19}) \div (9.1 \times 10^{-31})}$

$\cong 6 \times 10^6 \frac{m}{s} = 6000 \frac{km}{s}$

Exercise #2 $E = h\nu$

$= h\left(\frac{c}{\lambda}\right)$ because $c = \nu\lambda$

$= h\left(\frac{c}{h/p}\right)$ because $\lambda = h/p$

$= pc$ Q.E.D.

Exercise #3 Let $R_\odot$ be the distance from the earth to the sun. The area of the sphere with the sun at the centre and the earth at the surface is $4\pi R_\odot$. In one second, the amount of energy crossing this sphere is $1350 \times 4\pi R_\odot^2$. Let m be the mass this represents.

$\Rightarrow E = 1350 \times 4\pi R_\odot^2 = mc^2$

$\Rightarrow m = c^{-2} \times 1350 \times 4\pi R_\odot^2$. (Now, $R_\odot \cong 150 \times 10^6$ km $= 150 \times 10^9$ m)

$= (3 \times 10^8)^{-2} \times 1350 \times 4\pi \times (150 \times 10^9)^2$

$= 4.3$ million tons.

**Exercise #4** $F = \frac{dp}{dt}$, where $F \equiv$ force
$p \equiv$ momentum
$t \equiv$ time

$\Rightarrow F\,dt = dp$

$\Rightarrow F\int dt = \int dp +$ constant

$\Rightarrow Ft = p +$ constant

At $t=0, p=0 \Rightarrow$ constant $=0$

$\Rightarrow Ft = p = mv = m_0\gamma v \equiv \frac{m_0 v}{\sqrt{1-v^2/c^2}}$

$\Rightarrow Ft = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$. Now solve for $v$...

$\Rightarrow F^2t^2\left(1-\frac{v^2}{c^2}\right) = m_0^2 v^2$

$\Rightarrow F^2t^2 - F^2t^2v^2/c^2 = m_0^2 v^2$

$\Rightarrow F^2t^2 = v^2(m_0^2 + F^2t^2/c^2)$

$\Rightarrow Ft = v\sqrt{m_0^2 + F^2t^2/c^2}$

$\Rightarrow cFt = v\sqrt{m_0^2c^2 + F^2t^2}$

$\Rightarrow c = v\sqrt{m_0^2c^2/(F^2t^2) + 1}$

$\Rightarrow v = \frac{c}{\sqrt{1+\left(\frac{m_0 c}{Ft}\right)^2}}$ Q.E.D.

**Exercise #5** Upon emitting the burst of light, the "left" mass suffers a mass change from $m_1$ to $m_1'$, acquiring a velocity of $v_1$. The light burst carries momentum $P$ satisfying $E = pc \Rightarrow P = E/c$. Thus the recoil velocity $v_1$ must be such that momentum is conserved: $0 =$ initial momentum $= \frac{E}{c} + m_1'v_1 =$ final momentum. $\Rightarrow v_1 = \frac{-E/c}{m_1'}$. If $m_1$ was initially located at $x=0$, then its position at any later time is $x_1(t) = v_1 t = -Et/m_1'c$. Now, when the light burst slams into $m_2$, $m_2 \to m_2'$ and the mass picks up a momentum equal to that of the light pulse, i.e.:

$M_2' V_2 = p = E/c$. Solving for $V_2$, we see that $V_2 = E/(M_2' c)$. Therefore the position of the second block is $x_2(t) = \underbrace{L}_{\text{because the block is initially stationary at } x = L} + V_2 (t - \underbrace{\tfrac{L}{c}}_{\text{time taken for light pulse to reach } M_2})$. 

$$\Rightarrow x_2(t) = L + \frac{E}{M_2' c}\left(t - \frac{L}{c}\right), \quad t \geq \frac{L}{c}.$$

Let the total mass of the system be $M$, and let the position of the centre of mass be $\bar{x}$ before the radiation was emitted from $M_1$ and $\bar{x}'$ after it was absorbed by $M_2$. Then:

$$M\bar{x} = M_1 \cdot 0 + M_2 \cdot L$$

$$\begin{aligned} M\bar{x}' &= M_1' x_1(t) + M_2' x_2(t) \\ &= M_1'\left(\frac{-Et}{M_1' c}\right) + M_2'\left(L + \frac{E}{M_2' c}\left(t - \frac{L}{c}\right)\right) \\ &= \cancel{-\frac{Et}{c}} + M_2' L + \cancel{\frac{Et}{c}} - \frac{EL}{c^2} \\ &= L\left(M_2' - \frac{E}{c^2}\right) \end{aligned}$$

The centre of mass of the isolated system cannot have moved, i.e. $M\bar{x} = M\bar{x}'$

$$\Rightarrow M_2 \cdot \cancel{L} = \cancel{L}\left(M_2' - \frac{E}{c^2}\right)$$

$$\Rightarrow \frac{E}{c^2} = M_2' - M_2 \equiv \text{mass "}m\text{" of light}$$

$$\Rightarrow E = mc^2 \quad \text{Q.E.D.}$$

Exercise # 6 Fractional error $\equiv$ "$\varepsilon$"

$$\equiv \frac{|K_{relativistic} - K_{Newtonian}|}{K_{relativistic}}$$

$$= \frac{\left|\cancel{M_0} c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) - \frac{1}{2}\cancel{M_0} v^2\right|}{\cancel{M_0} c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)}$$

$$\varepsilon = \left| \frac{\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) - \frac{1}{2}\frac{v^2}{c^2}}{\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) - 1} \right| = \left| 1 - \frac{1}{2} \frac{v^2/c^2}{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1} \right|$$

For cases (i) − (iii), our calculators will have trouble evaluating $\varepsilon$. Since $v \ll c$ for these cases, we might be tempted to use $(1+\theta)^a \approx 1 + a\theta$, $\|\theta\| \ll 1$, to simplify the above expressions. Unfortunately, in this context the binomial approximation is too crude:

$$\varepsilon = \left| 1 - \frac{1}{2} \frac{v^2/c^2}{\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1} \right| \approx \left| 1 - \frac{1}{2} \frac{v^2/c^2}{1 + \frac{v^2}{2c^2} - 1} \right| = 0.$$

What we need in this context, is the more sensitive approximation: $(1+\theta)^a \approx 1 + a\theta + \frac{1}{2}a(a-1)\theta^2$, $\|\theta\| \ll 1$.
Hence, for cases (i) − (iii), we may write:

$$\varepsilon \approx \left| 1 - \frac{1}{2} \frac{v^2/c^2}{1 + \frac{v^2}{2c^2} + \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{v^4}{c^4} - 1} \right| = \left| 1 - \frac{1}{2} \frac{1}{\frac{1}{2} + \frac{3}{8}\frac{v^2}{c^2}} \right|$$

$$= \left| 1 - \frac{1}{1 + \frac{3}{4}\frac{v^2}{c^2}} \right| = \left| 1 - \left(1 + \frac{3}{4}\frac{v^2}{c^2}\right)^{-1} \right| \approx \left| 1 - \left(1 - \frac{3}{4}\frac{v^2}{c^2}\right) \right|$$

$$= \frac{3}{4}\frac{v^2}{c^2}.$$

(i) Here, $v/c = \frac{3}{3 \times 10^8} = 10^{-8} \Rightarrow \varepsilon \cong \frac{3}{4}\left(\frac{v}{c}\right)^2 = 0.75 \times 10^{-16}$

(ii) $v/c = 10^{-6} \Rightarrow \varepsilon \cong \frac{3}{4}\left(\frac{v}{c}\right) = 0.75 \times 10^{-12}$

(iii) $v/c = \frac{10^4}{3 \times 10^8} = \frac{1}{3} \times 10^{-4} \Rightarrow \varepsilon \cong \frac{3}{4}\left(\frac{v}{c}\right)^2 = \frac{3}{4} \times \frac{1}{9} \times 10^{-8}$

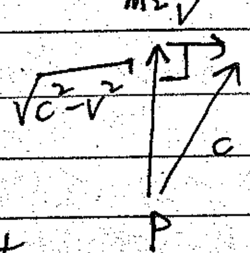
$$\cong 10^{-9}$$

(iv) Use formula at top of page, with $\frac{v}{c} = 0.1$

$$\Rightarrow \varepsilon = \left| 1 - \frac{1}{2} \frac{0.01}{\frac{1}{\sqrt{1-0.1^2}} - 1} \right| = \left| 1 - \frac{1}{2} \frac{0.01}{\frac{1}{\sqrt{.99}} - 1} \right| = 7.5 \times 10^{-3}$$

we could have used the same approx. formula as in (i) − (iii) to obtain this!!

(v) $\varepsilon = \left| 1 - \frac{1}{2} \frac{0.81}{\frac{1}{\sqrt{1-0.81}} - 1} \right| \cong 0.7.$




**Exercise #7** In going from $P$ to $M_2$, the light must be aimed into the wind at such that the resultant velocity is along $PM_2$ (see diagram above). Thus, by Pythagoras' theorem, the resultant velocity relative to the interferometer is $\sqrt{c^2-v^2}$. This velocity is reversed upon reflection from $M_2$. Thus the total time $t_2$ taken for the light to travel from $P$ to $M_2$ and back again is:

$$t_2 = \frac{\text{distance}}{\text{speed relative to interferometer}}$$

$$= \frac{2l_2}{\sqrt{c^2-v^2}} = \frac{2l_2/c}{\sqrt{1-v^2/c^2}}. \quad \text{Q.E.D.}$$

**Exercise #8** Equations (25) and (26) of the lecture:

$$(25)\ t_1 = \frac{2l_1/c}{1-\frac{v^2}{c^2}} \ ; \ (26)\ \frac{2l_2/c}{\sqrt{1-\frac{v^2}{c^2}}} = t_2.$$

If we rotate the interferometer, then:

- $t_1 \to t_1' =$ formula (26) with $l_2$ replaced by $l_1$
$$= \frac{2l_1/c}{\sqrt{1-v^2/c^2}}.$$

and: • $t_2 \to t_2' =$ formula (25) with $l_1$ replaced by $l_2$
$$= \frac{2l_2/c}{1-v^2/c^2}.$$

$$\Rightarrow \Delta' = t_1' - t_2'$$

$$= \frac{2l_1/c}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{2l_2/c}{1-\frac{v^2}{c^2}}$$

$$= \frac{2l_1}{c}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - \frac{2l_2}{c}\left(1-\frac{v^2}{c^2}\right)^{-1}$$

$$\simeq \frac{2l_1}{c}\left(1+\frac{v^2}{2c^2}\right) - \frac{2l_2}{c}\left(1+\frac{v^2}{c^2}\right) = \frac{2}{c}(l_1-l_2) + \frac{v^2}{c^3}(l_1-2l_2).$$

Q.E.D.

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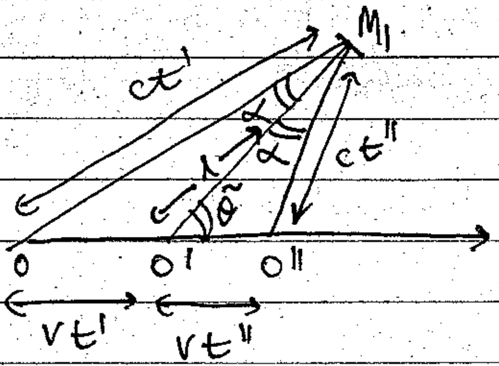
## Exercise #9




The diagram shows what things look like from the rest frame of the wind. $\theta + \tilde\theta = \pi/2$, with $\theta$ as defined in Figure 9 of the notes. Apply the cosine rule to triangle $0'M_1 0''$, to get:

$$c^2 t''^2 = l^2 + v^2 t''^2 - 2lvt''\cos\tilde\theta$$

$$\Rightarrow 0 = (c^2 - v^2)t''^2 + 2lvt''\cos\tilde\theta - l^2.$$

This quadratic in $t''$ may be solved for $t''$ using quadratic formula $\Rightarrow$

$$t'' = \frac{-2lv\cos\tilde\theta \pm \sqrt{4l^2v^2\cos^2\tilde\theta + 4(c^2-v^2)l^2}}{2(c^2-v^2)}$$

(choose "+" since $t''>0$)

$$= \frac{-lv\cos\tilde\theta + l\sqrt{v^2(\cos^2\tilde\theta - 1) + c^2}}{c^2 - v^2}$$

($\sin^2\tilde\theta + \cos^2\tilde\theta = 1$ $\therefore \cos^2\tilde\theta - 1 = -\sin^2\tilde\theta$)

$$= \left(-lv\cos\tilde\theta + l\sqrt{c^2 - v^2\sin^2\tilde\theta}\right)/(c^2 - v^2)$$

~~Similarly, apply the cosine rule to triangle $0M_10'$, then use the quadratic formula to solve for $t'$, keeping only the root that gives $t'>0$, to obtain: $t' =$~~

Now apply cosine rule to triangle $0M_10'$:

$$c^2t'^2 = v^2t'^2 + l^2 - 2vt'l\cos(\pi - \tilde\theta)$$

$$= v^2t'^2 + l^2 + 2vt'l\cos\tilde\theta$$

($\cos(\pi - \tilde\theta) = \cos\pi\cos\tilde\theta + \sin\pi\sin\tilde\theta = -\cos\tilde\theta$)

Solve this quadratic for $t'$ to get:

$$t' = \frac{lv\cos\tilde\theta + l\sqrt{c^2 - v^2\sin^2\tilde\theta}}{c^2 - v^2}$$

Now add together our formulae for $t'$ and $t''$:

$$t' + t'' = \frac{2l\sqrt{c^2 - v^2\sin^2\tilde\theta}}{c^2 - v^2}$$ = time taken to go from origin to mirror $M_1$ and back.

If we add $\frac{\pi}{2}$ radians to $\tilde\theta$, we will obtain:

$$\tilde t' + \tilde t'' = \frac{2l\sqrt{c^2 - v^2\sin^2(\tilde\theta + \frac{\pi}{2})}}{c^2 - v^2}$$ = time taken to go from origin to mirror $M_2$ and back.

Now, to answer the question,

~~$t' + t'' - (\tilde t' + \tilde t'')$~~

Now, to answer the question:

$6\frac{1}{2}$

$$\Delta t(\tilde{\theta}) \equiv \tilde{t}' + \tilde{t}'' - (t' + t'')$$

$$= \frac{2l}{c^2 - v^2}\left(\sqrt{c^2 - v^2 \sin^2(\tilde{\theta} + \frac{\pi}{2})} - \sqrt{c^2 - v^2 \sin^2 \tilde{\theta}}\right)$$

$$= \frac{2lc}{c^2 - v^2}\left(\sqrt{1 - \frac{v^2}{c^2}\sin^2(\tilde{\theta} + \frac{\pi}{2})} - \sqrt{1 - \frac{v^2}{c^2}\sin^2 \tilde{\theta}}\right)$$

$$\approx \frac{2lc}{c^2 - v^2}\left(1 - \frac{v^2}{2c^2}\sin^2(\tilde{\theta} + \frac{\pi}{2}) - \left\{1 - \frac{v^2}{2c^2}\sin^2 \tilde{\theta}\right\}\right)$$

$$= \frac{2lc}{c^2 - v^2} \times \frac{v^2}{2c^2} \times \left(\underbrace{\sin^2 \tilde{\theta}}_{\left(\frac{1}{2}(1 - \cos 2\tilde{\theta})\right)} - \underbrace{\sin^2(\tilde{\theta} + \frac{\pi}{2})}_{\left(\frac{1 - \cos(2(\tilde{\theta} + \frac{\pi}{2}))}{2} = \frac{1}{2}(1 - \cos(2\tilde{\theta} + \pi)) = \frac{1}{2}(1 + \cos 2\tilde{\theta})\right)}\right)$$

$$= \frac{\cancel{2}lc}{c^2 - v^2} \times \frac{v^2}{2c^{\cancel{2}}} \times \cancel{\frac{1}{2}}(\cancel{1} - \cos 2\tilde{\theta} - \cancel{1} - \cos 2\tilde{\theta})$$

$$= \frac{lv^2}{\cancel{2}c(c^2 - v^2)} \times -\cancel{2}\cos 2\tilde{\theta}$$

$$= -\frac{lv^2 \cos 2\tilde{\theta}}{c(c^2 - v^2)} \approx -\frac{lv^2 \cos 2\tilde{\theta}}{c^3}$$

Now, $\theta + \tilde{\theta} = \pi/2 \Rightarrow \tilde{\theta} = \pi/2 - \theta$

$$\Rightarrow \Delta t(\theta) = -\frac{lv^2}{c^3}\cos\left(2\left(\frac{\pi}{2} - \theta\right)\right)$$

$$= -\frac{lv^2}{c^3}\cos(\pi - 2\theta)$$

$$= -\frac{lv^2}{c^3}\cos\pi \cos(-2\theta)$$

$$= \frac{lv^2}{c^3}\cos 2\theta$$

Q.E.D.

Exercise #10 The time taken for the light pulse to travel from the presiding observer at $(0,0,0)$ to $(x,y,z)$ and back again is:

$$\Delta t = \frac{\text{distance to } (x,y,z) \text{ and back again}}{\text{Speed of signal}}$$

$$= \frac{2\sqrt{x^2+y^2+z^2}}{c}$$

$\Rightarrow$ If we multiply $\Delta t$ by $\frac{1}{2}c$, we obtain the distance to the particle at $(x,y,z)$. Q.E.D.

Exercise #11 Evidently, the "forward" Lorentz transformations (56) can be written in matrix form as:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c^2} & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

This matrix is of the form $\begin{pmatrix} \underline{\underline{A}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{I}} \end{pmatrix}$ where

$$\underline{\underline{A}} \equiv \begin{pmatrix} \gamma & -\frac{\gamma v}{c^2} \\ -\gamma v & \gamma \end{pmatrix}$$

$$\underline{\underline{0}} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underline{\underline{I}} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, it is easy to show, by direct calculation, that

$$\begin{pmatrix} \underline{\underline{A}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{I}} \end{pmatrix}^{-1} = \begin{pmatrix} \underline{\underline{A}}^{-1} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{I}} \end{pmatrix}.$$

Now, by using the usual formula for inverting a $2\times2$ matrix,* we see that:

$$\underline{\underline{A}}^{-1} = \frac{1}{\underbrace{\gamma^2 - (\gamma v)\left(\frac{\gamma v}{c^2}\right)}_{\text{this is unity}}} \begin{pmatrix} \gamma & \frac{\gamma v}{c^2} \\ \gamma v & \gamma \end{pmatrix}.$$

Therefore the inverse Lorentz transformations are:

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \frac{\gamma v}{c^2} & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}.$$ QED

* i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

This concludes our "brute force" derivation of the inverse Lorentz transformations from the forward Lorentz transformations.

A faster way to do this is by making use of the "v-reversal symmetry" derived in the next exercise. Using this symmetry, together with the fact that $\gamma(v) = \gamma(-v)$, we may write down the inverse Lorentz transformations on inspection, given the forward Lorentz transformations.

Exercise #12 The inertial frames ~~S~~ $S$ and $S'$, in standard configuration, may be represented by the sketch in Figure 1.

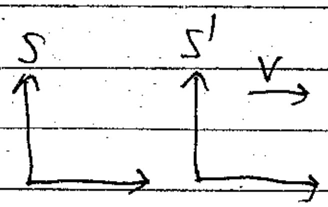


Figure 1

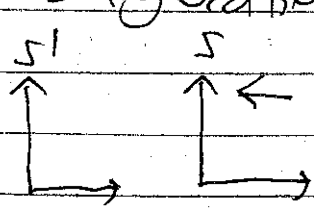


Figure 2

If we apply the v-reversal to this sketch, we obtain Figure 2. Yet, by the principle of relativity, Figures 1 and 2 describe exactly the same physical situation. (Explicitly, if frame $S'$ moves to the right with speed $v$ relative to the rest frame of $S$ (Fig. 1), then this is equivalent to frame $S$ moving to the left with speed $v$ relative to the rest frame of $S'$ (Fig 2.)). Hence the validity of applying a v-reversal to any relativistic transformation formula which relates unprimed and ~~pri~~ primed quantities (from $S$ and $S'$ respectively).

Exercise #13 (a) $c\sqrt{\gamma^2 - 1} = c\sqrt{\frac{1}{1-\frac{v^2}{c^2}} - 1} = c\sqrt{\frac{1-\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}}}$

$= c\sqrt{\frac{v^2/c^2}{1-v^2/c^2}} = \frac{v}{\sqrt{1-v^2/c^2}} = \gamma v$ Q.E.D.

(b) $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \dfrac{d\gamma}{dv} = \dfrac{d}{dv}\left(1-\dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}$ 9

$$= -\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}} \frac{d}{dv}\left(-\frac{v^2}{c^2}\right)$$ chain rule

$$= \cancel{-}\frac{1}{\cancel{2}}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\cancel{-}\frac{\cancel{2}v}{c^2}\right)$$

$$= \frac{\gamma^3 v}{c^2} \Rightarrow c^2 d\gamma = \gamma^3 v\, dv$$

Q.E.D.

(c) $\dfrac{d}{dv}(\gamma v) = \gamma \dfrac{d}{dv}(v) + v\dfrac{d\gamma}{dv}$ ... product rule

$$= \gamma + v\frac{d\gamma}{dv}$$

$$= \gamma + v\left(\frac{\gamma^3 v}{c^2}\right) \quad \text{... from part (b)}$$

$$= \gamma^3\left(\gamma^{-2} + \frac{v^2}{c^2}\right) = \gamma^3\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right) = \gamma^3$$

$\Rightarrow d(\gamma v) = \gamma^3 dv$ Q.E.D.

<u>Exercise #14</u> Suppose the burst of light originates from $(x, y, z) = (0, 0, 0)$ at time $t = 0$. Then, at time $t > 0$, the spherical shell of light will be a sphere of radius $ct$. Therefore,

$$\sqrt{x^2+y^2+z^2} = ct$$
$$x^2+y^2+z^2 = c^2t^2$$
$$0 = c^2t^2 - x^2 - y^2 - z^2 \quad \text{Q.E.D.}$$

for events $(t, x, y, z)$ which lie on the expanding light burst.

We now Lorentz transform from $S$ to $S'$, where the inertial frames $S$ and $S'$ are in the standard configuration. Thus, from the Lorentz transformations (57), we see that the equation describing the burst in $S'$ is:

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$$0 = c^2\gamma^2(t' + vx'/c^2)^2 - \gamma^2(x' + vt')^2 - y'^2 - z'^2$$
$$= c^2\gamma^2(t'^2 + v^2x'^2c^{-4} + 2t'vx'c^{-2})$$
$$\quad - \gamma^2(x'^2 + v^2t'^2 + 2x'vt') - y'^2 - z'^2$$
$$= \gamma^2[c^2t'^2 + v^2x'^2c^{-2} + \cancel{2t'vx'} - x'^2 - v^2t'^2 - \cancel{2x'vt'}] - y'^2 - z'^2$$
$$= \gamma^2[t'^2(c^2 - v^2) + x'^2(v^2c^{-2} - 1)] - y'^2 - z'^2$$
$$= \gamma^2[t'^2c^2(1 - v^2c^{-2}) - x'^2(1 - v^2c^{-2})] - y'^2 - z'^2$$
$$= \gamma^2[t'^2c^2\gamma^{-2} - x'^2\gamma^{-2}] - y'^2 - z'^2$$
$$= c^2t'^2 - x'^2 - y'^2 - z'^2$$

Q.E.D.

This result, which at first sight appears to do violence to "common sense", is bound up with the relativity of simultaneity. Points which, as measured in S, are reached at the <u>same</u> time, are reached at <u>different</u> times as measured in S', in such a fashion that the light is properly described as lying on a spherical shell expanding at c in <u>both</u> frames.

<u>Exercise #15</u> $\Delta s^2 \equiv c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

Now, if we apply a "v-reversal" to equation (59) of the lectures we see that:

$$\Delta t = \gamma(\Delta t' + \frac{v\Delta x}{c^2})$$
$$\Delta x = \gamma(\Delta x' + v\Delta t)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$

Hence, $\Delta s^2 = c^2\gamma^2(\Delta t' + \frac{v\Delta x}{c^2})^2 - \gamma^2(\Delta x' + v\Delta t)^2 - \Delta y^2 - \Delta z^2$

$=$ [... steps are exactly analogous to those at top ... of this page]

$= c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$.

Q.E.D.

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Exercise #16 We separately prove properties (a) – (d) as listed under heading 10.6 of the notes.

We shall use the "matrix notation" introduced in exercise #1, and shall denote:

$$\underline{\Omega}(v) \equiv \begin{pmatrix} \gamma & \frac{-\gamma v}{c^2} & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\alpha)$$

(a) $\underline{\Omega}(v_1) \in G$ and $\underline{\Omega}(v_2) \in G$, where $G$ is the Lorentz group. Then, with $\gamma_1 \equiv \gamma(v_1)$ and $\gamma_2 \equiv \gamma(v_2)$:

$$(\beta) \quad \underline{\Omega}(v_1)\underline{\Omega}(v_2) = \begin{pmatrix} \gamma_1 & \frac{-\gamma_1 v_1}{c^2} & 0 & 0 \\ -\gamma_1 v_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_2 & \frac{-\gamma_2 v_2}{c^2} & 0 & 0 \\ -\gamma_2 v_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\gamma) \quad = \begin{pmatrix} \left(\gamma_1\gamma_2 + \frac{\gamma_1\gamma_2 v_1 v_2}{c^2}\right) & \left(\frac{-\gamma_1\gamma_2 v_2}{c^2} - \frac{\gamma_1\gamma_2 v_1}{c^2}\right) & 0 & 0 \\ \left(-\gamma_1\gamma_2 v_1 - \gamma_1\gamma_2 v_2\right) & \left(\frac{+\gamma_1\gamma_2 v_1 v_2}{c^2} + \gamma_1\gamma_2\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, motivated by the Einstein velocity addition formula*, let us introduce:

$$(\delta) \quad v_3 \equiv \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

We now work out each of the four elements in the top left corner of this matrix, writing them in terms of $\gamma_3$ and $v_3$.

* My apologies – this will not be introduced until the next chapter – sorry about that!

FIRST ELEMENT

$$\begin{pmatrix} \bullet & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

$$\gamma_1\gamma_2 + \frac{\gamma_1\gamma_2 v_1 v_2}{c^2}$$

$$= \gamma_1\gamma_2\left(1+\frac{v_1v_2}{c^2}\right)$$

$$= \frac{1+v_1v_2/c^2}{\sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}}$$

$$= \left(\frac{(1-v_1^2/c^2)(1-v_2^2/c^2)}{(1+v_1v_2/c^2)^2}\right)^{-\frac{1}{2}}$$

$$= \left(\frac{(1+v_1v_2/c^2)^2 - (1+v_1v_2/c^2)^2 + (1-v_1^2/c^2)(1-v_2^2/c^2)}{\left(1+\frac{v_1v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}}$$

$$= \left(1 - \frac{(1+v_1v_2/c^2)^2 - (1-v_1^2/c^2)(1-v_2^2/c^2)}{\left(1+\frac{v_1v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}}$$

$$= \left(1 - \frac{\cancel{1} + \cancel{\frac{v_1^2v_2^2}{c^4}} + \frac{2v_1v_2}{c^2} \cancel{-1} + v_1^2/c^2 + v_2^2/c^2 - \cancel{\frac{v_1^2v_2^2}{c^4}}}{\left(1+\frac{v_1v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}}$$

$$= \left(1 - \frac{v_1^2+v_2^2+2v_1v_2}{c^2\left(1+\frac{v_1v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}}$$

$$= \left(1 - \frac{(v_1+v_2)^2}{c^2\left(1+\frac{v_1v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}} \equiv \left(1-\frac{v_3^2}{c^2}\right)^{-\frac{1}{2}} \equiv \gamma_3 \qquad (\varepsilon)$$

SECOND ELEMENT

$$\begin{pmatrix} \times & \times & \times & \times \\ \bullet & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

$$-\gamma_1\gamma_2 v_1 - \gamma_1\gamma_2 v_2$$

$$= -\gamma_1\gamma_2(v_1+v_2)$$

$$= \frac{-(v_1+v_2)}{\sqrt{(1-v_1^2/c^2)(1-v_2^2/c^2)}}$$

$$= -\frac{(v_1+v_2)}{1+\frac{v_1v_2}{c^2}} \cdot \frac{1+\frac{v_1v_2}{c^2}}{\sqrt{\left(1-\frac{v_1^2}{c^2}\right)\left(1-\frac{v_2^2}{c^2}\right)}} \equiv -v_3\frac{1+\frac{v_1v_2}{c^2}}{\sqrt{\left(1-\frac{v_1^2}{c^2}\right)\left(1-\frac{v_2^2}{c^2}\right)}}$$

$\equiv v_3$, by $(\delta)$

$$= -v_3 \left( \frac{(1 - v_1^2/c^2)(1 - v_2^2/c^2)}{\left(1 + \frac{v_1 v_2}{c^2}\right)^2} \right)^{-1/2}$$

look at the fourth and last line of calculation (ε), to see that this is equal to $\gamma_3$.

$$= -\gamma_3 v_3 \qquad (\phi)$$

THIRD ELEMENT

$$\begin{pmatrix} \times & \bullet & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

This is the same as the second element divided by $c^2$. Hence:

$$-\frac{\gamma_1 \gamma_2 v_2}{c^2} - \frac{\gamma_1 \gamma_2 v_1}{c^2} = -\frac{\gamma_3 v_3}{c^2} \qquad (\gamma)$$

FOURTH ELEMENT

$$\begin{pmatrix} \times & \times & \times & \times \\ \times & \bullet & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

Same as first element.

(η)

Collecting together these results for each of the elements, we see that equation (γ) becomes:

$$\underline{\underline{\Lambda}}(v_1)\,\underline{\underline{\Lambda}}(v_2) = \begin{pmatrix} \gamma_3 & -\gamma_3 v_3/c^2 & 0 & 0 \\ -\gamma_3 v_3 & \gamma_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \underline{\underline{\Lambda}}(v_3) \in G.$$

⇒ Group axiom (a) is satisfied.

(b) Evidently, $I \equiv \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$ is such that, 14

$\forall$ ("for all") $a \in G$, $aI = aI = a$.
$\Rightarrow$ Group axiom (b) is satisfied.

(c) In exercise #1, we gave two different derivations of the inverse Lorentz transformation.
$\Rightarrow \forall a \in G \; \exists a^{-1} \mid aa^{-1} = a^{-1}a = I.$

"there exists" "such that"

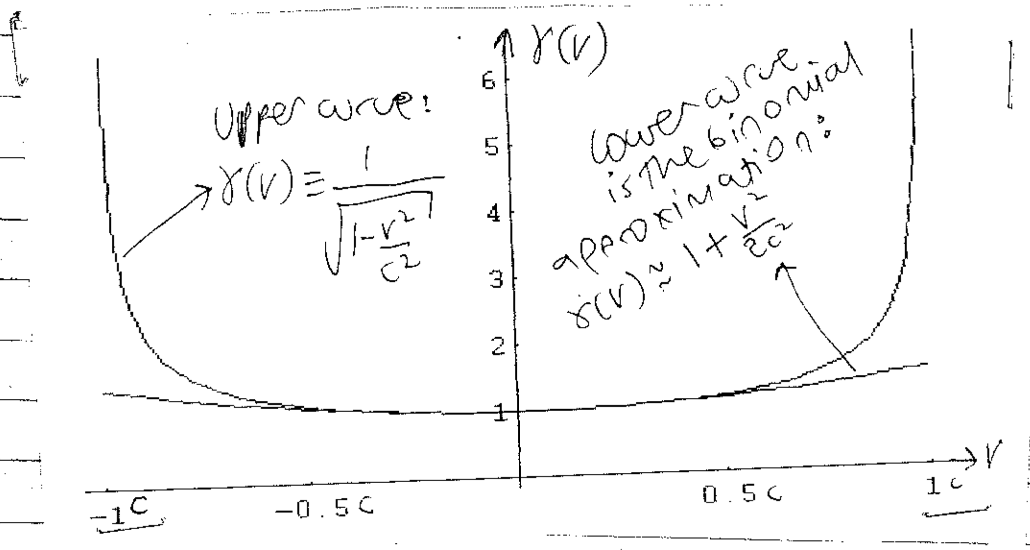
$\Rightarrow$ group axiom (c) is satisfied.

(d) Products of 4x4 matrices are associative; so:
$\forall a, b, c \in G$, $(ab)c = a(bc)$.
$\Rightarrow$ Group axiom (d) is satisfied.

Since all four group axioms are satisfied, we conclude that the Lorentz transformations form a group, known as the Lorentz group.

PS: Can you think of a better way to derive (a)??

Exercise #17



**Exercise #18** Denote by $(x, t_1)$ and $(x, t_2)$ the two events that occur at the same point in some inertial reference frame $S$. Also, denote by $\Delta \equiv t_2 - t_1$ the time separation of these events in $S$. Now introduce a moving frame $S'$ which is in standard configuration with $S$. The transformed value of $\Delta$ is evidently:

$$\begin{aligned}\Delta' &\equiv t_2' - t_1' \\ &= \gamma(v)\left(t_2 - \frac{vx}{c^2}\right) - \gamma(v)\left(t_1 - \frac{vx}{c^2}\right) \\ &= \gamma(v)(t_2 - t_1) \\ &\equiv \gamma(v)\Delta\end{aligned}$$

Hence the ratio of $\Delta'$ and $\Delta$ is a positive quantity, namely $\gamma(v)$. Therefore $\Delta$ and $\Delta'$ are either (i) both positive or (ii) both negative, for any value of the boost velocity $v$, provided $-c < v < c$. Therefore the temporal order of the events $(x, t_1)$ and $(x, t_2)$ is the same for all reference frames.

Since $\gamma(v) \geq 1$, $\gamma(v=0) = 1$, we see that the least time separation to the two events is assigned in $S$.

**Exercise #19** Denote by $(x_1, t)$ and $(x_2, t)$ the two events which occur at the same time in some inertial frame $S$. Denote by $\Delta$ their time separation; obviously, in $S$, $\Delta \equiv t_{\text{event }2} - t_{\text{event }1} = t - t = 0$. Now Lorentz transform to $S'$, with the "usual" "standard configuration". Then the transformed value of the

$$\Delta' \equiv t_2' - t_1'$$
$$= \gamma(v)\left(t - \frac{vx_2}{c^2}\right) - \gamma(v)\left(t - \frac{vx_1}{c^2}\right)$$
$$= \gamma(v)\frac{v}{c^2}(x_1 - x_2)$$

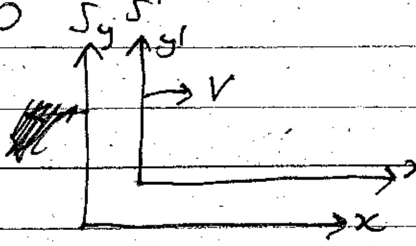
As $v \to c$, $\Delta'$ evidently becomes arbitarily large, provided $x_1 \neq x_2$. Therefore there is no limit on the time separation assigned to our two events in other inertial frames.

Now we look at the space separation of the two events. Let $\Delta x = x_2 - x_1$. Then, in $S'$, we have:

$$\Delta x' \equiv x_2' - x_1'$$
$$= \gamma(v)(x_1 - vt) - \gamma(v)(x_2 - vt)$$
$$= \gamma(v)(x_1 - x_2)$$

Hence, provided $x_1 \neq x_2$, the space separation of the events varies from infinity (as $v \to \pm c$) to a minimum which is measured in $S$.

Exercise #20




In frame $S'$, the points $(x', y')$ of the rod obey:

$$y' = ut'.$$

Now we use the Lorentz transformation (56) to transform this equation back to $S$:

$$y = u\gamma\left(t - \frac{vx}{c^2}\right)$$

$$y = -\frac{vu\gamma}{c^2}x + u\gamma t.$$ At any time $t$, this is a straight line.

Gradient of straight line $= \frac{dy}{dx} = -\frac{vu\gamma}{c^2} = \tan\theta$, where $\theta$ is the angle of inclination of the rod to the positive x axis in S. Therefore:

$$\theta = \tan^{-1}\left(\frac{-vu\gamma}{c^2}\right) = -\tan^{-1}\left(\frac{vu\gamma}{c^2}\right)$$ Q.E.D.

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Exercise #21 With reference to Figure 12, and relative to an observer on the ground, the light must travel a progressively longer path for each tick, the faster the clock is moving. When the clock is stationary, one tick involves the light traveling a distance of $2l_0$; when the clock moves, one tick involves the light travelling the path ABC, which is greater than $2l_0$. Therefore this clock, when moving, "runs slow" with respect to the observer with respect to which the clock is moving.

Exercise #22 Begin with formula (86):

$$(86)\quad (c^2-u'^2) = \frac{c^2(c^2-u^2)(c^2-v^2)}{(c^2-u_1v)^2}$$

If $u' < c \Rightarrow u'^2 < c^2 \Rightarrow c^2 - u'^2 > 0$ and therefore the left side of (86) is $> 0$. Now, $v < c \Rightarrow v^2 < c^2 \Rightarrow c^2 - v^2 > 0$, therefore the second bracket in the numerator of the right side of (86) is $>0$. Also, the factors $c^2$ and $(c^2-u_1v)^2$ are both evidently $>0$.

Since 3 out of the four bracketed quantities of (86) are $>0$, and $c^2 > 0$, we conclude that the remaining bracketed quantity is $>0$. Thus: $c^2 - u^2 > 0 \Rightarrow u^2 < c^2 \Rightarrow u < c$. Q.E.D. Interpretation: the vectorial sum of two sub-luminal velocity vectors yields a subliminal result.

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Exercise #23 Divide both sides of (86a) by $c^2$, to give:

$$1-\frac{u'^2}{c^2} = \frac{(c^2-u^2)(c^2-v^2)}{(c^2-u_1v)^2} \times \frac{c^{-4}}{c^{-4}}$$

$$1-\frac{u'^2}{c^2} = \frac{(1-u^2/c^2)(1-v^2/c^2)}{(1-u_1v/c^2)^2}$$

Now take the reciprocal of both sides of the equation, and then take the square root of everything:

$$\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{\left|1-\frac{u_1v}{c^2}\right|}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}$$

Now, from eq. (55c) of the lectures:

$$\gamma(v) \equiv \frac{1}{\sqrt{1-v^2/c^2}}$$

and so our equation becomes:

$$\gamma(u') = \frac{|1-u_1v/c^2|}{\gamma^{-1}(u)\gamma^{-1}(v)} \longrightarrow \text{modulus}$$

signs can be replaced with brackets since $u_1 < c$ and $v < c$ $\Rightarrow 1-\frac{u_1v}{c^2} > 0$.

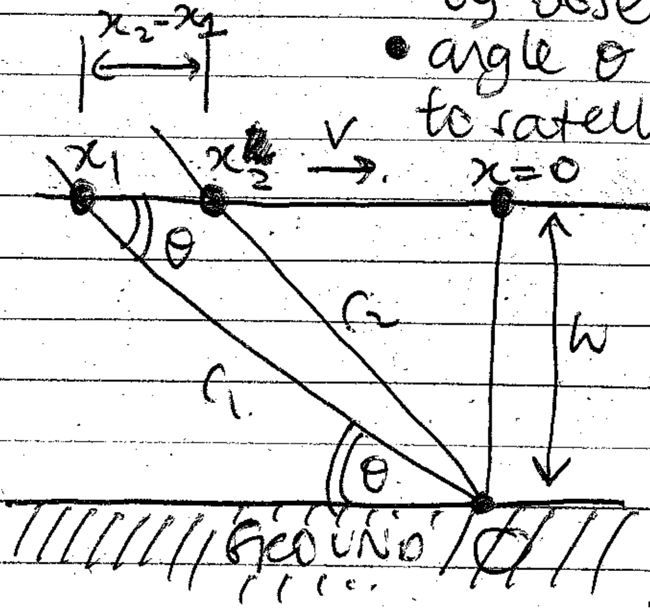
$$\Rightarrow \frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left(1-\frac{u_1v}{c^2}\right)$$

which is (87a). Equation (87b) results from applying a v-reversal transformation to (87a),

together with the fact that $\gamma(-v) = \gamma(v)$.
(Cf. Exercise #12.)

Exercise #24 Relevant parameters:

- velocity of satellite, $v$
- impact parameter, $h$
- frequency $\nu_0$ of emitted radiation in satellite rest frame
- received frequency $\nu'$ measured by observer on earth.
- angle $\theta$ between line of sight to satellite and the ground.
- distance from observer to satellite




The satellite moves from left to right, at speed $v$ and altitude $h$ above the surface of the earth. Consider two successive pulses emitted from positions $x_1$ and $x_2$ in the diagram. In the rest frame of the satellite, the interval between pulses is $1/\nu_0$; in the rest frame of the observer, this is stretched by time dilation:

(a) $$t_2' - t_1' = \frac{\gamma}{\nu_0}.$$

The first pulse takes $r_1/c$ seconds to reach the observer at O, while the second pulse takes $r_2/c$ seconds to reach the observer. Thus the time separation $\tau'$ between the pulses received by O is:

(b) $$\tau' = \left(t_2' + \frac{r_2}{c}\right) - \left(t_1' + \frac{r_1}{c}\right)$$

$$\Rightarrow \tau' = \underbrace{(t_2' - t_1')}_{(a)} + \frac{r_2 - r_1}{c}$$

$$= \gamma/\nu_0 + \frac{r_2 - r_1}{c}$$



(c) $$\equiv \gamma\tau_0 + \frac{r_2 - r_1}{c}$$

→ Period between pulses in the satellite rest frame.

Now, $r_2 - r_1 \simeq (x_2 - x_1)\cos\theta$

$$= v(t_2' - t_1')\cos\theta$$

$$= v\gamma\nu_0^{-1}\cos\theta$$

(d) $$\equiv v\gamma\tau_0\cos\theta$$

Hence (c) becomes:

$$\tau' \cong \gamma\tau_0 + v\gamma\tau_0\cos\theta/c$$

$$= \gamma\tau_0\left(1 - \frac{v\cos\theta}{c}\right)$$

$$\Rightarrow \frac{\tau'}{\tau_0} = \frac{\nu_0}{\nu'} = \gamma\left(1 - \frac{v}{c}\cos\theta\right)$$

(e) $$\Rightarrow \nu' = \frac{\nu_0\gamma^{-1}}{1 - \frac{v}{c}\cos\theta} = \frac{\nu_0\sqrt{1 - v^2/c^2}}{\underbrace{1 - \frac{v}{c}\cos\theta}}$$

Note that this could have been obtained directly from (12), with $u_r = -v\cos\theta$ and $u = v$!!

Now, $\cos\theta = \cos\theta(t)$

(f) $$= \frac{-vt}{\sqrt{h^2 + v^2t^2}}$$

$\Rightarrow$ (e) becomes: $$\boxed{\nu'(t) = \frac{\nu_0\sqrt{1 - v^2/c^2}}{1 + \frac{v^2 t}{c\sqrt{h^2 + v^2t^2}}}}$$

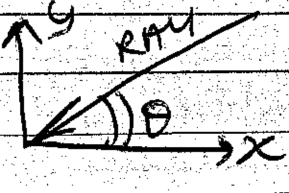
**Exercise #25**

$u_1 = -c\cos\theta$

$u_2 = -c\sin\theta$

$u_2' = -c\sin\theta'$

(a)




Now, from the second member of the velocity transformation equations (77) in the lecture notes, we have:

$$u_2' = \frac{u_2}{\gamma\left(1-\frac{u_1 v}{c^2}\right)} \Rightarrow -c\sin\theta' = \frac{-c\sin\theta}{\gamma(v)\left(1+\frac{v\cos\theta}{c}\right)}$$

Hence $$\boxed{\sin\theta' = \frac{\sin\theta}{\gamma(v)\left(1+\frac{v}{c}\cos\theta\right)}}$$ which is equation (95). Q.E.D.

(b) $$\frac{\sin\theta'}{1+\cos\theta'} = \frac{\sin(2\frac{1}{2}\theta')}{2\frac{1}{2}(1+\cos\theta')}$$

→ let $\alpha = \frac{\theta'}{2}$ in $\sin 2\alpha = 2\sin\alpha \times \cos\alpha$

→ let $\alpha = \frac{\theta'}{2}$ in $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$

$$= \frac{2\sin(\theta'/2)\cos(\theta'/2)}{2\cos^2(\theta'/2)}$$

$$= \tan(\theta'/2) \quad \text{Q.E.D.}$$

(c) $$\tan\left(\tfrac{1}{2}\theta'\right) = \frac{\sin\theta'}{1+\cos\theta'}$$ → use (95) (numerator), → use (94) (denominator)

$$= \frac{\left(\dfrac{\sin\theta}{\gamma(v)\left(1+\frac{v}{c}\cos\theta\right)}\right)}{1+\dfrac{\cos\theta+v/c}{1+\frac{v}{c}\cos\theta}}$$

$$= \frac{\sin\theta}{\gamma(v)\left(1+\frac{v}{c}\cos\theta+\cos\theta+\frac{v}{c}\right)}$$

$$= \frac{\sin\theta}{\gamma(v)\left(1+\frac{v}{c}\right)(1+\cos\theta)}$$

$$= \frac{\sqrt{1-v^2/c^2}}{1+v/c}\;\frac{\sin\theta}{1+\cos\theta} \longrightarrow \tan\left(\tfrac{1}{2}\theta\right), \text{ by } (96)$$

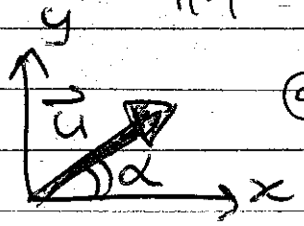
$$= \frac{\sqrt{(1-v/c)(1+v/c)}}{1+v/c}\tan\left(\tfrac{1}{2}\theta\right)$$

$$= \sqrt{\frac{1-v/c}{1+v/c}}\tan\left(\tfrac{1}{2}\theta\right) = \sqrt{\frac{c-v}{c+v}}\tan\left(\tfrac{1}{2}\theta\right).$$

Q.E.D.

21/3/2

Exercise #26




(a) $\begin{cases} u_1 = u\cos\alpha \\ u_2 = u\sin\alpha \\ u_1' = u'\cos\alpha' \\ u_2' = u'\sin\alpha' \end{cases}$

Write down the first two members of the velocity transformation formulae (77) in the notes:

(b) $u_1' = \dfrac{u_1 - v}{1 - \frac{u_1 v}{c^2}}$ (c) $u_2' = \dfrac{u_2}{\gamma\left(1 - \frac{u_1 v}{c^2}\right)}$

Now substitute from (a) into (b) & (c):

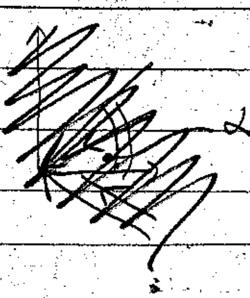
(d) $u'\cos\alpha' = \dfrac{u\cos\alpha - v}{1 - \frac{u\cos\alpha . v}{c^2}}$ (e) $u'\sin\alpha' = \dfrac{u\sin\alpha}{\gamma\left(1 - \frac{u\cos\alpha . v}{c^2}\right)}$

Now divide (e) by (d) so as to eliminate the irrelevant factor of $u'$:

$\rightarrow$ (f) $\tan\alpha' = \dfrac{u\sin\alpha\left(1 - \frac{uv\cos\alpha}{c^2}\right)}{\gamma\left(1 - \frac{uv\cos\alpha}{c^2}\right)(u\cos\alpha - v)}$

Cancel the common factor, and then divide both numerator and denominator by u, to arrive at the desired result.

Exercise #27 29/4/2



The desired ratio of solid angles, $\frac{d\Omega}{d\Omega'}$, is evidently equal to the ratio $\frac{d\alpha}{d\alpha'}$ of "ordinary" angles, squared.

(1) Thus: $\frac{d\Omega}{d\Omega'} = \left(\frac{d\alpha}{d\alpha'}\right)^2$. What remains is to calculate $\frac{d\alpha}{d\alpha'}$.

Take equation (95) from the lecture notes, replacing $\theta$ by $\alpha$:

(2) $$\sin\alpha' = \frac{\sin\alpha}{\gamma\left(1+\frac{v}{c}\cos\alpha\right)}$$

Hence:

$$\frac{d}{d\alpha}\sin\alpha' = \frac{d}{d\alpha}\,\frac{\sin\alpha}{\gamma\left(1+\frac{v}{c}\cos\alpha\right)}$$

$$\frac{d\alpha'}{d\alpha}\cdot\frac{d}{d\alpha'}\sin\alpha' = \frac{d}{d\alpha}\,\frac{\sin\alpha}{\gamma\left(1+\frac{v}{c}\cos\alpha\right)}$$

(3) $$\left(\frac{d\alpha'}{d\alpha}\right)\cos\alpha' = \frac{1}{\gamma}\cdot\underbrace{\frac{\left(1+\frac{v}{c}\cos\alpha\right)\cos\alpha + \sin\alpha\cdot\frac{v}{c}\cdot\sin\alpha}{\left(1+\frac{v}{c}\cos\alpha\right)^2}}_{\text{we used quotient rule of differentiation.}}$$

Hence,

$$\left(\frac{d\alpha'}{d\alpha}\right)\cos\alpha' = \frac{1}{\gamma}\,\frac{\cos\alpha + \frac{v}{c}(\cos^2\alpha + \sin^2\alpha)}{\left(1+\frac{v}{c}\cos\alpha\right)^2}$$

$$④ \quad = \frac{1}{\gamma}\,\frac{\cos\alpha + v/c}{\left(1+\frac{v}{c}\cos\alpha\right)^2}$$

Therefore:

note negative sign!!

$$⑤ \quad \frac{d\omega}{d\omega'} = \left(\frac{d\alpha'}{d\alpha}\right)^{-2} = \left(\frac{\cos\alpha + \frac{v}{c}}{\gamma.\cos\alpha'.\left(1+\frac{v}{c}\cos\alpha\right)^2}\right)^{-2}$$

Now, the required formula is entirely in terms of $\alpha, v, c$ ⇒ we need to eliminate $\alpha'$ from the formula above. This is done using equation (94) from the lecture notes, with $\theta$ replaced by $\alpha$:

$$⑥ \quad \cos\alpha' = \frac{\cos\alpha + v/c}{1+\frac{v}{c}\cos\alpha}.$$

If we substitute ⑥ into ⑤, we get the required result:

$$\frac{d\omega}{d\omega'} = \left(\frac{\cos\alpha + v/c}{\gamma.\left(\frac{\cos\alpha + v/c}{1+\frac{v}{c}\cos\alpha}\right)\left(1+\frac{v}{c}\cos\alpha\right)^2}\right)^{-2}$$

$$= \gamma^2(v)\left(1+\frac{v}{c}\cos\alpha\right)^2 \qquad \text{Q.E.D.}$$