Exercise #28 We see from equation (A.9) of Rindler's appendix on tensors, that $g_{\mu\nu}$ and $g^{\mu\nu}$, when considered as matrices, must multiply together to give the unit matrix. This holds true if the metric tensor $g_{\mu\nu}$ is numerically equal to its conjugate $g^{\mu\nu}$:

$$\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}. \quad \text{Q.E.D.}$$

Exercise #29 From equations (110) in the notes, we see that:

$$P^{0'}_0 \equiv \frac{\partial x^{0'}}{\partial x^0} = \gamma, \quad P^{0'}_1 \equiv \frac{\partial x^{0'}}{\partial x^1} = -\frac{\gamma v}{c}, \quad P^{0'}_2 = 0, \quad P^{0'}_3 = 0$$

$$P^{1'}_0 \equiv \frac{\partial x^{1'}}{\partial x^0} = -\frac{\gamma v}{c}, \quad P^{1'}_1 \equiv \frac{\partial x^{1'}}{\partial x^1} = \gamma, \quad P^{1'}_2 = 0, \quad P^{1'}_3 = 0$$

$$P^{2'}_0 = 0, \quad P^{2'}_1 = 0, \quad P^{2'}_2 = 1, \quad P^{2'}_3 = 0$$

$$P^{3'}_0 = 0, \quad P^{3'}_1 = 0, \quad P^{3'}_2 = 0, \quad P^{3'}_3 = 1$$

Similarly, from equations (111), one can obtain (113b).

Also, $T^{1'2'} = P^{1'}_\mu P^{2'}_\nu T^{\mu\nu}$ ... transformation law for rank-two contravariant tensor

$$\begin{aligned} = & P^{1'}_0 P^{2'}_0 T^{00} + P^{1'}_0 P^{2'}_1 T^{01} + P^{1'}_0 P^{2'}_2 T^{02} + P^{1'}_0 P^{2'}_3 T^{03} \\ & + P^{1'}_1 P^{2'}_0 T^{10} + P^{1'}_1 P^{2'}_1 T^{11} + P^{1'}_1 P^{2'}_2 T^{12} + P^{1'}_1 P^{2'}_3 T^{13} \\ & + P^{1'}_2 P^{2'}_0 T^{20} + P^{1'}_2 P^{2'}_1 T^{21} + P^{1'}_2 P^{2'}_2 T^{22} + P^{1'}_2 P^{2'}_3 T^{23} \\ & + P^{1'}_3 P^{2'}_0 T^{30} + P^{1'}_3 P^{2'}_1 T^{31} + P^{1'}_3 P^{2'}_2 T^{32} + P^{1'}_3 P^{2'}_3 T^{33} \end{aligned}$$

this row is zero since $P^{1'}_3 = 0$
" " " " " $P^{1'}_2 = 0$

these two columns are zero since $P^{2'}_0 = P^{2'}_1 = 0$

we are therefore left with the four terms in the top right corner.

$$\Rightarrow T^{1'2'} = P^{1'}_0 P^{2'}_2 T^{02} + P^{1'}_0 \underbrace{P^{2'}_3}_{\text{zero}} T^{03} + P^{1'}_1 P^{2'}_2 T^{12} + P^{1'} \underbrace{P^{2'}_3}_{\text{zero}} T^{13}$$

$$= \left(-\frac{\gamma v}{c}\right)(1)T^{02} + (\gamma)(1)T^{12}$$

$$= \gamma\left(T^{12} - vT^{02}/c\right) \quad \text{Q.E.D.}$$

Exercise #30 Start with (119):

$$A = \gamma\left(\underbrace{\frac{d\gamma}{dt}}_{\text{one in rest frame}} c, \underbrace{\frac{d\gamma}{dt}\vec{u}}_{\text{zero in rest frame}} + \gamma\vec{a}\right) \overset{\text{rest}}{\underset{\text{frame}}{=}} \left(\frac{d\gamma}{dt}c, \vec{a}\right)$$

Now, $\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

$$= -\frac{1}{2}\cdot\left(1-\frac{v^2}{c^2}\right)^{-3/2}\cdot\frac{-2v}{c^2}$$

$$= 0 \text{ when } v = 0$$

$\Rightarrow$ in the rest frame, $A = (0, \vec{a})$

Exercise #31 (a) From (118), $U = \gamma(c, u)$. The temporal component γc of this four-vector is $\geq c$ since $\gamma \geq 1$. Thus this temporal component never vanishes; therefore U never vanishes.

(b) $U^2 = \gamma^2 c^2 - \gamma^2 u^2$. Since this is an invariant, we can calculate its value in any inertial frame: choose the rest frame. Therefore $U^2 = c^2$, and so the square of the four velocity is always the same, viz. c^2.
A less elegant proof is as follows:

$$U^2 = \gamma^2 c^2 - \gamma^2 u^2 = \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\right)^2 (c^2 - u^2)$$

$$= \frac{1}{1-u^2/c^2} \times (c^2 - u^2)$$

$$= \frac{c^2}{c^2 - u^2} \times (c^2 - u^2) = c^2 \quad \text{Q.E.D.}$$

(c) In exercise #30, we showed that $\mathbf{A}=(0,\mathbf{a})$ in the rest frame of the particle. Now, $\mathbf{A}^2$ has the same value in all frames, so we may as well evaluate it in the rest frame. $\Rightarrow \mathbf{A}^2=(0,\mathbf{a})\cdot(0,\mathbf{a})=0^2-a^2=-a^2$ $\equiv -\alpha^2$

(d) $$\mathbf{U}\cdot\mathbf{A} = \overbrace{\gamma(c,\vec{0})}^{\mathbf{U}\text{ in rest frame}}\cdot\underbrace{(0,\vec{a})}_{\mathbf{A}\text{ in rest frame}} = 0$$

(d) (119) $\Rightarrow \mathbf{A}=\gamma(\dot{\gamma}c, \dot{\gamma}\vec{u}+\gamma\vec{a})$, $\dot{\gamma}\equiv d\gamma/dt$.

$\Rightarrow \mathbf{A}^2=\gamma^2(\dot{\gamma}^2c^2-(\dot{\gamma}\vec{u}+\gamma\vec{a})^2)$ Q.E.D.

(e) From (c) and (d) above,

$$\alpha^2=-\mathbf{A}^2=\gamma^2((\dot{\gamma}\vec{u}+\gamma\vec{a})^2-\dot{\gamma}^2c^2)$$
$$=\gamma^2(\dot{\gamma}^2u^2+\gamma^2a^2+2\gamma\dot{\gamma}\vec{u}\cdot\vec{a}-\dot{\gamma}^2c^2)$$
$$=\gamma^2(\dot{\gamma}^2(u^2-c^2)+2\gamma\dot{\gamma}\vec{u}\cdot\vec{a}+\gamma^2a^2)$$
$$=\gamma^2(\dot{\gamma}^2(u^2-c^2)+2\gamma\dot{\gamma}u\frac{du}{dt}+\gamma^2a^2)$$

Q.E.D. 24/5/12

Exercise #32 We will write the temporal component of a given four-vector $\mathbf{U}$ as U^0, and the spatial component as $\vec{u}$. Thus $\mathbf{U}\equiv(U^0,\vec{u})$, and:

$\mathbf{T}\equiv(T^0,\vec{T})$, $\mathbf{T}^2\equiv(T^0)^2-\vec{T}^2>0$ (time like)
$\mathbf{S}\equiv(S^0,\vec{S})$, $\mathbf{S}^2\equiv(S^0)^2-\vec{S}^2<0$ (space like)
$\mathbf{N}\equiv(N^0,\vec{N})$, $\mathbf{N}^2\equiv(N^0)^2-\vec{N}^2=0$ (null vector)
c/f bottom right corner of p. 26 of the notes
$\mathbf{V}\equiv(V^0,\vec{V})$, $\mathbf{V}^2\equiv(V^0)^2-\vec{V}^2$ (general four-vector)

(i) We need to show that $\mathbf{V}\cdot\mathbf{T}=0 \Rightarrow \mathbf{V}^2<0$. Since $\mathbf{V}\cdot\mathbf{T}$ is an invariant, we can evaluate it in any inertial frame. We change the inertial reference frame to a "primed" frame where $\mathbf{T}'\equiv(T^{0\prime},\vec{0})$ and $\mathbf{V}'=(V^{0\prime},\vec{V}')$. Then $0=\mathbf{V}'\cdot\mathbf{T}'=T^{0\prime}V^{0\prime}$. Since $T^{0\prime}\neq 0 \Rightarrow V^{0\prime}=0 \Rightarrow \mathbf{V}'^2\equiv(V^{0\prime})^2-\vec{V}'^2=0^2-\vec{V}'^2<0 \Rightarrow \mathbf{V}$ is an S, i.e. V is spacelike. Q.E.D.

Next, we need to show that $\mathbf{V}\cdot\mathbf{N} = 0 \Rightarrow \mathbf{V}^2 < 0$.

Transform to a primed inertial frame such that:

$\mathbf{V}' = (V^{0'}, \vec{V}')$

$\mathbf{N}' = (N^{0'}, N^{0'}, 0, 0)$ this can always be done by appropriately rotating axes so that two of the spatial components vanish. The remaining nonzero spatial component must equal the transformed temporal component, since $\mathbf{N}'^2 = 0$.

Hence, in the primed frame, $\mathbf{V}'\cdot\mathbf{N}' = 0 \Rightarrow V^{0'}N^{0'} - V'^{x}N^{0'} = 0$.

$\Rightarrow$ cancel the $N^{0'}$ and obtain $V^{0'} = V'^{x}$. Thus:

$$\mathbf{V}'^2 = (V^{0'})^2 - \vec{V}'^2 \equiv \underbrace{(V^{0'})^2 - (V^{x})^2}_{0,\ \text{since } V^{0'} = V^{x'}} - (V^{y})^2 - (V^{z})^2$$

$$\Rightarrow \mathbf{V}'^2 = -(V^{y})^2 - (V^{z})^2 < 0$$

$\Rightarrow \mathbf{V}'$ is an S. Q.E.D.

(ii) Let $\mathbf{T}_{(1)} \equiv (T^0_{(1)}, \vec{T}_{(1)})$, $\mathbf{T}_{(2)} \equiv (T^0_{(2)}, \vec{T}_{(2)})$. Since both $\mathbf{T}_{(1)}$ and $\mathbf{T}_{(2)}$ are isochronous, the numbers $T^0_{(1)}$ and $T^0_{(2)}$ both have the same sign in a given inertial frame. The sum $\tilde{\mathbf{T}}$ of $\mathbf{T}_{(1)}$ and $\mathbf{T}_{(2)}$ evidently has a temporal component $T^0_{(1)} + T^{0'}_{(2)}$ which has the same sign as both $\mathbf{T}_{(1)}$ and $\mathbf{T}_{(2)}$. Therefore $\mathbf{T}_{(1)}, \mathbf{T}_{(2)}$ and $\tilde{\mathbf{T}}$ are isochronous. We now need to complete the proof by showing that $\tilde{\mathbf{T}}$ is timelike, i.e. that $\tilde{\mathbf{T}}^2 > 0$. Now,

$$\tilde{T}^2 = (T^0_{(1)} + T^0_{(2)}, \vec{T}_{(1)} + \vec{T}_{(2)})^2$$

$$= (T^0_{(1)})^2 + (T^0_{(2)})^2 + 2T^0_{(1)}T^0_{(2)} - (\vec{T}_{(1)})^2 - (\vec{T}_{(2)})^2 - 2\vec{T}_{(1)}\cdot\vec{T}_{(2)}$$

$$= (\mathbf{T}_{(1)})^2 + (\mathbf{T}_{(2)})^2 + 2(T^0_{(1)}T^0_{(2)} - \vec{T}_{(1)}\cdot\vec{T}_{(2)})$$

Now transform to a frame where $\vec{T}_{(1)}' = \vec{0}$. Thus:

$$\mathbf{T}^2 = (\mathbf{T}_{(1)})^2 + (\mathbf{T}_{(2)})^2 + 2[T^{0'}_{(1)}T^{0'}_{(2)}]$$

Each of the three terms on the right side of this equation are greater than zero, and so $\mathbf{T}^2 > 0$, which completes the proof.

- $(\mathbf{T}_{(1)})^2$ and $(\mathbf{T}_{(2)})^2$ are both > 0 because they are time-like vectors
- $T^{0'}_{(1)} T^{0'}_{(2)} > 0$ because the fact that $\mathbf{T}'_{(1)}$ and $\mathbf{T}'_{(2)}$ are isochronous implies that their spatial components $T^{0'}_{(1)}$ & $T^{0'}_{(2)}$ have the same sign.

Now, we need to show that the sum of a T and an N is a T which is isochronous with them. $\mathbf{T}+\mathbf{N} \equiv (T^0+N^0, \vec{T}+\vec{N})$. Evidently $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{T}+\mathbf{N}$ are isochronous because all will have the same sign for their temporal component. To complete the proof, we need to show that $\mathbf{T}+\mathbf{N}$ is time like, i.e. that $(\mathbf{T}+\mathbf{N})^2 > 0$.

$$(\mathbf{T}+\mathbf{N})^2 = \mathbf{T}^2 + \underset{\substack{\parallel \\ 0}}{\mathbf{N}^2} + 2\,\mathbf{N}\cdot\mathbf{T}$$

$$= \mathbf{T}^2 + 2\,\underbrace{(N^0 T^0 - \vec{N}\cdot\vec{T})}$$

Now transform to a frame where $\vec{T}' = \vec{0}$

$$= \mathbf{T}^2 + 2\,N^{0'} T^{0'}$$

> 0 since $\mathbf{T}$ is timelike

> 0 since $N^{0'}$ and $T^{0'}$ have the same sign (i.e. they are isochronous, by assumption)

> 0 Q.E.D.

<u>Exercise # 33</u> Zero-component lemma. Let the four vector $\mathbf{N} \equiv (N^0, N^1, N^2, N^3)$.

First, assume that the temporal component N^0 of $\mathbf{N}$ is zero for all frames. Then the transformation law for $\mathbf{N}$ is the relevant

Lorentz transformation formulae:

$$\begin{cases} u^{0\prime} = \gamma(u^0 - vu^1/c) \\ u^{1\prime} = \gamma(u^1 - vu^0/c) \\ u^{2\prime} = u^2 \\ u^{3\prime} = u^3 \end{cases}$$

with $u^0 = 0$, ie:

$$\begin{cases} 0 = -\gamma v u^1/c \Rightarrow \underline{\underline{u^1 = 0}} \\ u^{1\prime} = \gamma u^1 \\ u^{2\prime} = u^2 \\ u^{3\prime} = u^3 \end{cases}$$

So now we see that both u^0 and u^1 are zero in all frames, i.e. $u = (0, 0, u^2, u^3)$.

I will refer to this later!

Thus the fact that u^0 vanishes implies that u^1 vanishes, as can be seen by doing a Lorentz transform in the x direction. If we now Lorentz transform in the y direction, we will see that $u^2 = 0$; a Lorentz transform in the z direction will show that $u^3 = 0$. We conclude that if the temporal component of a four-vector vanishes in all frames, then so do its spatial components.

To complete the proof, we evidently need to show that, if any spatial component vanishes in all inertial frames, then so do the other components. Now, since we can always rotate our xyz axis, we need only show that, if the x component of a four vector vanishes in all inertial frames, this implies that all components vanish in all frames. Therefore, suppose that u is such that:

$$u = (u^0, 0, u^1, u^2)$$

x component vanishes in all inertial frames.

Then, with $u^1 = u^1 = 0$, the Lorentz transformation formulae becomes:

$$\begin{cases} u^{0\prime} = \gamma u^0 \\ 0 = -\gamma v u^0/c \\ u^{2\prime} = u^2 \\ u^{3\prime} = u^3 \end{cases} \quad \Rightarrow u^0 = 0 \Rightarrow u^{0\prime} = 0$$

$\Rightarrow u^0$ vanishes in all frames.

Hence the temporal component vanishes in all frames. But we showed earlier (see argument which follows the smiley face) that this in turn implies that all components vanish in all frames.

This completes our proof of the zero-component lemma.

<u>Exercise #34</u> From equation (128) of the notes, we know that $\mathbf{P} = (mc, \vec{p})$. Suppose, then, that $\mathbf{P}_1 = (m_1 c, \vec{p}_1)$ and $\mathbf{P}_2 = (m_2 c, \vec{p}_2)$. Then $\mathbf{P}_1 \cdot \mathbf{P}_2 = m_1 m_2 c^2 - \vec{p}_1 \cdot \vec{p}_2$. (a).
The expression $\mathbf{P}_1 \cdot \mathbf{P}_2$ is invariant, i.e. it has the same value in all inertial frames. However, the terms $m_1, m_2, \vec{p}_1, \vec{p}_2$ on the right side of (a) will change from frame to frame. In a given frame, $m_1 = \gamma(v_1) m_{01}$, where v_1 is the speed of particle #1 in that frame and m_{01} is the rest mass of particle #1. Similarly, in a given frame, $m_2 = \gamma(v_2) m_{02}$, where v_2 is the speed of particle #2 in that frame and m_{02} is the rest mass of particle #2.

If we evaluate the right side of (a) in the rest frame of particle #1, where $\vec{p}_1 = 0$, and $m_1 \to m_{01}$, then:

$$(b)\quad \mathbf{P}_1 \cdot \mathbf{P}_2 = m_{01} m_2 c^2 - \underbrace{\vec{p}_1}_{=0} \cdot \vec{p}_2 = m_{01} m_2 c^2$$

Similarly, if we evaluate the invariant $\mathbf{P}_1 \cdot \mathbf{P}_2$ in the rest frame of particle #2, where $\vec{p}_2 = 0$ and $m_2 \to m_{02}$, then:

$$(c)\quad \mathbf{P}_1 \cdot \mathbf{P}_2 = m_1 m_{02} c^2 - \vec{p}_1 \cdot \underbrace{\vec{p}_2}_{=0} = m_1 m_{02} c^2$$

From (a), (b), (c) we see that:

$$(d)\quad \mathbf{P}_1 \cdot \mathbf{P}_2 = c^2 m_{01} m_2 = c^2 m_1 m_{02}$$

But $m_1 = m_{01}\gamma(v)$, where v is the speed of the first particle in the rest frame of the second. But this speed v is just their relative speed v_{rel}. Hence this last term can be written as $m_{01}\gamma(v_{rel}) m_{02} c^2$, and (d) becomes equation (134) of the notes.

Q.E.D.

Exercise #35 Begin with equation (87b) from the notes, which shows how the γ-factor of a particle transforms between a pair of inertial frames which are in the usual standard configuration with respect to one another: $\frac{\gamma(u)}{\gamma(u')} = \gamma(v)\left(1+\frac{u_1'v}{c^2}\right)$. Here, u and u' are the speed of the said particle in S and S' respectively, u_1' is the x'-component of the particle's velocity in S', and v is the relative speed of the two inertial frames. With $u' = u_1' = v$ and $u = V$, the γ-transformation formula becomes:

$$\frac{\gamma(V)}{\gamma(v)} = \gamma(v)\left(1+\frac{v^2}{c^2}\right) \Rightarrow \gamma(V) = \gamma^2(v)\left(1+\frac{v^2}{c^2}\right)$$

$$= -\gamma^2(v)\left(-1-\frac{v^2}{c^2}\right)$$

$$= -\gamma^2(v)\left(-2+1-\frac{v^2}{c^2}\right)$$

$$= -\gamma^2(v)\left(-2+\gamma^{-2}(v)\right)$$

$$= 2\gamma^2(v) - 1$$

$$\Rightarrow \frac{\gamma(V)+1}{2} = \gamma^2(v) \Rightarrow \frac{1}{\gamma^2(v)} = \frac{2}{\gamma(V)+1} \quad \text{Q.E.D.}$$

Exercise #36 In the non-relativistic limit, $\gamma(V) \to 1$ in equation (139) of the notes, and so this equation becomes (a) $\tan\theta \tan\phi = \frac{2}{1+1} = 1$. Now, formula (a) implies that (b) $\tan\theta = \frac{1}{\tan\phi} \equiv \cot\phi$. Put this formula to one side for the moment, and recall the double angle formula (c) $\tan(\lambda-\phi) = \frac{\tan\lambda - \tan\phi}{1+\tan\lambda\tan\phi}$.

Let $\lambda \to \frac{\pi}{2}$, either from above or below; when $\lambda \to \frac{\pi}{2}$, $\|\tan\lambda\| \to \infty$ and so formula (c) becomes: (d) $\lim_{\lambda\to\frac{\pi}{2}} \tan(\lambda-\phi) = \lim_{\lambda\to\frac{\pi}{2}} \frac{\tan\lambda}{\tan\lambda\tan\phi}$

$= \cot\phi$.

Therefore $\tan(\frac{\pi}{2}-\phi) = \cot\phi$. Thus (b) becomes: (e) $\tan\theta = \tan(\frac{\pi}{2}-\phi)$, and so (f) $\theta = \frac{\pi}{2} - \phi + n\pi$, where n is an integer. This may be rearranged to give: (g) $\theta+\phi=\frac{\pi}{2}+n\pi$. The left side of this expression is the opening angle, which must lie between 0 and π. Hence the integer $n=0$, and (g) becomes: (h) $\theta+\phi=\frac{\pi}{2}$, $V \ll c$, which is the desired result. Q.E.D.

Our next task is to show that, for the relativistic case, $\theta+\phi<\pi/2$. Now, (i) $\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\left(\dfrac{2}{\gamma(V)+1}\right)}$.

Now, $\gamma(V)>1 \Rightarrow \gamma(V)+1>2 \Rightarrow 0 \leq \dfrac{2}{\gamma(V)+1} < 1$

$$\Rightarrow 1 \leq 1+\frac{2}{\gamma(V)+1} < 2$$

... and so the denominator on the right side of (i) is strictly positive. Also, since $\theta \in [0,\frac{\pi}{2}]$ (see Figure 21) $\Rightarrow \tan\theta \geq 0$; similarly, $\phi \in [0,\frac{\pi}{2}] \Rightarrow \tan\phi \geq 0$; therefore the numerator of the right side of (i) is ≥ 0. Since $(\tan\theta+\tan\phi)\geq 0$ & $1-\left(\dfrac{2}{\gamma(V)+1}\right)>0 \Rightarrow \dfrac{\tan\theta+\tan\phi}{1-\left(\dfrac{2}{\gamma(V)+1}\right)} \geq 0$,

which means that (i) implies: (j) $\tan(\theta+\phi)\geq 0$. Since, from Figure 21, $\theta+\phi \leq \pi \Rightarrow (\theta+\phi)\in[0,\pi/2]$, because this will ensure that $\tan(\theta+\phi)\geq 0$. Since $\theta+\phi=\pi/2$ has already been shown to be the nonrelativistic limit, $\theta+\phi \in [0,\frac{\pi}{2})$ for the relativistic case. Q.E.D.

Exercise #37 Measure θ and ϕ, and then re-arrange (139) of the notes to give:

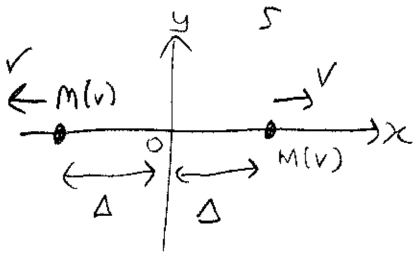
$$\gamma(V)=\frac{2}{\tan\theta\tan\phi}-1. \text{ Since } \gamma(V)=\frac{1}{\sqrt{1-V^2/c^2}}, \Rightarrow \frac{1}{\sqrt{1-V^2/c^2}}=\frac{2}{\tan\theta\tan\phi}-1.$$

Solve the formula for V to give:

$$V = c\sqrt{1 - \frac{1}{\left(\frac{2}{\tan\theta\tan\phi} - 1\right)^2}},$$

which will yield V from the know values of θ, ϕ. Note that this formula becomes useless if $\theta + \phi = 90°$!!

Exercise # 38



In the sketch we see two particles, of identical rest mass m_0, moving away from the origin in opposite directions, along the x-axis of some inertial frame S. These particles have equal speeds and are at all times equidistant from the origin O. The relativistic mass of both particles is $m_0\gamma(V) = m$ for both particles. The centre of mass of the system, according to someone is S, lies exactly midway between the two particles.

Now move to a frame of reference which is nailed to the right particle. In this frame, the mass of the right particle will be less than the mass of the left particle, and therefore the centre of mass will be closer to the left particle than it is to the right particle.

Exercise # 39 The incident pair of protons in the reaction $p + p \rightarrow p + p + \pi^0$ carries three-momentum. Therefore, by the conservation of three-momentum, the post-collision particles p, p, π^0 cannot be at rest, and so a part of the incident kinetic energy must remain as kinetic energy.

Exercise # 40 (a) $k = \dfrac{\text{rest mass energy of created particle}}{\text{kinetic energy of bullet}}$

m_0 = rest mass of created particle

M_0 = rest mass of bullet

m = relativistic mass of created particle

M = relativistic mass of bullet

$$= \frac{\text{rest mass energy of created particle}}{\text{energy of bullet} - \text{rest mass energy of bullet}}$$

$$= \frac{m_0 c^2}{Mc^2 - M_0 c^2}$$

$$= \frac{m_0 c^2}{M_0 \gamma c^2 - M_0 c^2} = \frac{m_0}{M_0(\gamma - 1)}.$$

Now use equation (149) for the threshold γ factor:

$$\Rightarrow k = \frac{m_0}{M_0(\gamma-1)} = \frac{m_0}{M_0\left(\underbrace{1 + \frac{2m_0}{M_0} + \frac{m_0^2}{2M_0^2}}_{\gamma_1 \text{ from } (149)} - 1\right)}$$

$$= \frac{m_0}{M_0\left(\frac{2m_0}{M_0} + \frac{m_0^2}{2M_0^2}\right)}$$

$$= \frac{m_0}{2m_0 + \frac{m_0^2}{2M_0}} = \frac{1}{2 + \frac{m_0}{2M_0}} = \frac{2}{4 + \frac{m_0}{M_0}} \quad \text{Q.E.D.}$$

(b) let M_0 = rest mass of electron/positron, and m_0 = rest mass of ψ. Then let $M_{01} = M_{02} = M_0$ and $\bar{m}_{cm} = m_0$ in (147), to give: $2M_0^2 + 2M_0^2\gamma = \bar{m}_{cm}^2$. At threshhold, $\bar{m}_{cm}^2 = m_0^2$ (remember, the electron and antielectron have annihilated) $\Rightarrow 2M_0^2 + 2M_0^2\gamma(V_{thresh.}) = m_0^2$.

$$\Rightarrow \gamma(V_{thresh}) = \frac{m_0^2 - 2M_0^2}{2M_0^2}$$

Denoting $\gamma(v_{thresh}) \equiv \gamma$ = threshold gamma, we have:

$$\gamma - 1 = \frac{m_0^2 - 2M_0^2}{2M_0^2} - 1 = \frac{m_0^2}{2M_0^2} - 2.$$ Put this into the formula

for the efficiency K that we derived earlier, namely

$$K = \frac{m_0}{M_0(\gamma-1)}, \text{ to get: } K = \frac{m_0}{M_0\left(\frac{m_0^2}{2M_0^2} - 2\right)} = \frac{1}{\frac{m_0}{2M_0} - \frac{2M_0}{m_0}} = \frac{2}{\frac{m_0}{M_0} - \frac{4M_0}{m_0}},$$

which is the required result. Since $m \approx 3700 M_0$, we have:

$$K \approx \frac{2}{3700 - (4/3700)} \approx \frac{2}{3700} = \frac{1}{1850},$$ which is miniscule.

To make the method almost 100% efficient, the method of "clashing" (or "colliding") beams can be used. Here, both target and bullet particles are accelerated to high energy (e.g. electrons and positrons can be accelerated in the same synchroton, in opposite directions), then accumulated in magnetic "storage rings", before being finally loosed on one another head-on. No energy need be wasted, since both the laboratory and centre-of-momentum frame now coincide: all the kinetic energy can be used to create new matter.

Exercise # 41 Take (15b) from the lecture notes as a start:

(a) $\mathbf{P}\cdot\mathbf{P}' = \mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')$. With $\mathbf{Q} = \gamma m(1,\vec{u})$, $\mathbf{P} = h\nu(1,\vec{n})$ and $\mathbf{P}' = h\nu'(1,\vec{n}')$, (a) becomes:

$$h\nu'(1,\vec{n}')\cdot h\nu(1,\vec{n}) = \gamma m(1,\vec{u})\cdot(h\nu - h\nu', h\nu\vec{n} - h\nu'\vec{n}')$$

$$h^2\nu\nu'(1-\vec{n}'\cdot\vec{n}) = \gamma m(h\nu - h\nu' - \vec{u}\cdot(h\nu\vec{n} - h\nu'\vec{n}'))$$

(b) $$h^{\cancel{2}}\nu\nu'(1-\vec{n}'\cdot\vec{n}) = \gamma m\cancel{h}(\nu - \nu' - \nu\vec{u}\cdot\vec{n} + \nu'\vec{u}\cdot\vec{n}')$$

since $\vec{u}\cdot\vec{n} = -u$, $\vec{n}\cdot\vec{n}' = -1$, $\vec{u}\cdot\vec{n}' = u$ for a head-on collision, (b) becomes:

$$2h\nu\nu' = \gamma m(\nu - \nu' + \nu u + \nu' u) = \gamma m\nu(1+u) - \gamma m\nu'(1-u)$$

Next, solve for $h\nu'$:

$$2\nu(h\nu') = \gamma m\nu(1+u) - (h\nu')\frac{\gamma m}{h}(1-u)$$

$$h\nu'\left[2\nu + \frac{\gamma m}{h}(1-u)\right] = \gamma m\nu(1+u)$$

$$\Rightarrow h\nu' = \frac{\gamma m\nu(1+u)}{2\nu + \frac{\gamma m}{h}(1-u)} = \frac{\gamma m(1+u)}{2+\frac{\gamma m}{h\nu}(1-u)} \quad \text{Q.E.D.}$$

With a view to interpreting this result, suppose that $u \approx 1$ (incoming particle travels near speed of light, in units where $c=1$) and that $\gamma \gg 1$. Then the above formula becomes:

$$\text{(c)} \quad h\nu' \approx \frac{2\gamma m}{2+\frac{\gamma m}{h\nu}(1-u)} = \frac{\gamma m}{1+\frac{\gamma m}{2h\nu}(1-u)} = \frac{\gamma m}{1+\frac{m}{2h\nu}(\gamma-\gamma u)}, \quad \gamma \gg 1.$$

need to be careful, since $\gamma \gg 1$ and $1-u \approx 0$!!

Now, from equation (a) of exercise #13, we have, in units where $c=1$, $\gamma u = \sqrt{\gamma^2-1}$. Hence $\gamma u = \sqrt{\gamma^2(1-\gamma^{-2})} = \gamma(1-\gamma^{-2})^{\frac{1}{2}}$. Since $\gamma \gg 1$, $\gamma^{-2} \ll 1$ and so we can use the binomial approximation on the square root to give $\gamma u \approx \gamma(1-\frac{1}{2}\gamma^{-2}) = \gamma - \frac{1}{2}\gamma^{-1}$. Thus $\gamma - \gamma u \approx \frac{1}{2}\gamma^{-1}$ when $\gamma \gg 1$. This allows us to write down the following approximation to (c):

$$\text{(d)} \quad h\nu' = \frac{m\gamma}{1+\frac{m}{2h\nu}(\gamma-\gamma u)} \approx \frac{m\gamma}{1+\frac{m}{2h\nu}\frac{1}{2}\gamma^{-1}} = \frac{m\gamma}{1+\frac{m}{4h\nu\gamma}}, \quad \gamma \gg 1.$$

Now consider the case of a cosmic ray proton ($\gamma \approx 10^{11}$, $m \approx 10^9\,eV$) colliding with a photon from the cosmic microwave background ($h\nu \approx 3\times10^{-4}\,eV$). From (d), the energy of the scattered photon is:

$$h\nu' \approx \frac{m\gamma}{1+\frac{m}{4h\nu\gamma}} \cong \frac{10^9 10^{11}}{1+\frac{10^9}{4(3\times10^{-4})10^{11}}} \cong \frac{10^{20}}{1+\frac{10^9}{10^1 10^{-4} 10^{11}}} = \frac{10^{20}}{1+\frac{10^9}{10^8}} \approx 10^{19}\,eV.$$

The scattered photon has therefore experienced a monumental gain in energy.

Exercise #42 From (161), we see that the four-force $\mathbf{F}$ has the components:

(a) $F^0 = \gamma(u) c^{-1} \frac{dE}{dt}$, $F^1 = \gamma(u) f^x$, $F^2 = \gamma(u) f^y$, $F^3 = \gamma(u) f^z$,

where u is the speed of the particle in a given frame and (f_x, f_y, f_z) are the Cartesian components of the three-force. Now, the components of the four-force transform just like space-time events under a Lorentz transformation, and so we can look at (110) and write down the transformation formulae for the components of the four-force:

$$\begin{cases} F^{0\prime} = \gamma(v)\left(F^0 - \frac{v}{c} F^1\right) \\ F^{1\prime} = \gamma(v)\left(F^1 - \frac{v}{c} F^0\right) \Rightarrow \gamma(u') f^{x\prime} = \gamma(v)\left(\gamma(u) f^x - \frac{v}{c}\gamma(u) c^{-1}\frac{dE}{dt}\right) \\ F^{2\prime} = F^2 \Rightarrow \gamma(u') f^{y\prime} = \gamma(u) f^y \\ F^{3\prime} = F^3 \Rightarrow \gamma(u') f^z = \gamma(u) f^z \end{cases}$$

Hence:

$$\begin{cases} f^{x\prime} = \dfrac{\gamma(v)\gamma(u)}{\gamma(u')}\left(f^x - \dfrac{v}{c^2}\dfrac{dE}{dt}\right) \\ f^{y\prime} = \dfrac{\gamma(u)}{\gamma(u')} f^y \\ f^{z\prime} = \dfrac{\gamma(u)}{\gamma(u')} f^z \end{cases}$$

Next, remember from equation (87a) that $\gamma(u')/\gamma(u) = \gamma(v)\left(1 - \frac{u_1 v}{c^2}\right)$.
Also, $dE/dt = d(mc^2)/dt = c^2 dm/dt$. Thus our transformation formulae become:

$$\begin{cases} f^{x\prime} = \gamma(v)\dfrac{1}{\gamma(v)\left(1 - \frac{u_1 v}{c^2}\right)}\left(f^x - \dfrac{v}{c^2} c^2 \dfrac{dm}{dt}\right) = \dfrac{f^x - v\frac{dm}{dt}}{1 - u_1 v/c^2} \\ f^{y\prime} = \dfrac{1}{\gamma(v)\left(1 - \frac{u_1 v}{c^2}\right)} f^y \\ f^{z\prime} = \dfrac{1}{\gamma(v)\left(1 - \frac{u_1 v}{c^2}\right)} f^z \end{cases}$$

which are the required results.

Exercise #43 We have the potentials $(\Phi, \vec{A})$. From (164) and (166), we know that both the electric and magnetic fields can be derived from these potentials via: (a) $\vec{B} = \nabla \times \vec{A}$ and (b) $\vec{E} = -\nabla \Phi - c^{-1}\partial_t \vec{A}$. Now suppose that we make the gauge transformation: (c) $\vec{A} \to \vec{A}' = \vec{A} + \nabla \Lambda$, (d) $\Phi \to \Phi' = \Phi - c^{-1}\partial_t \Lambda$. So, now we have the new potentials $(\Phi', \vec{A}')$, and we need to show that they yield the same electric and magnetic fields.

(e) $$\vec{B}' = \nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla \Lambda) = \underbrace{\nabla \times \vec{A}}_{\vec{B}} + \underbrace{\nabla \times \nabla \Lambda}_{\vec{0}} = \vec{B}$$

(f) $$\vec{E}' = -\nabla \Phi' - c^{-1}\partial_t \vec{A}' = -\nabla(\Phi - c^{-1}\partial_t \Lambda) - c^{-1}\partial_t(\vec{A} + \nabla \Lambda)$$
$$= -\nabla \Phi + \cancel{c^{-1}\partial_t \nabla \Lambda} - c^{-1}\partial_t \vec{A} - \cancel{c^{-1}\partial_t \nabla \Lambda}$$
$$= -\nabla \Phi - c^{-1}\partial_t \vec{A} \overset{(b)}{=} \vec{E}$$

We see from (e) & (f) that the electric and magnetic fields are unchanged by the gauge transformation.

Exercise #44 We have a scalar and vector potential $(\Phi, \vec{A})$ which do not satisfy the Lorentz condition (172), i.e. (a) $\nabla \cdot \vec{A} + c^{-1}\partial_t \Phi = f \neq 0$, where f will be some function of position and time. We need to choose a gauge function Λ so that the gauge transformed potentials, obtained by (b) $\vec{A} \to \vec{A}' = \vec{A} + \nabla \Lambda$, (c) $\Phi \to \Phi' = \Phi - c^{-1}\partial_t \Lambda$, do satisfy the Lorentz condition: (d) $\nabla \cdot \vec{A}' + c^{-1}\partial_t \Phi' = 0$. Substitute (b) and (c) into (a) and then make use of (d):

$$\Rightarrow \nabla \cdot \vec{A} + c^{-1}\partial_t \Phi = f$$
$$\Rightarrow \nabla \cdot \underbrace{(\vec{A}' - \nabla \Lambda)}_{\vec{A},\ \text{by (b)}} + c^{-1}\partial_t \underbrace{(\Phi' + c^{-1}\partial_t \Lambda)}_{\Phi,\ \text{by (c)}} = f$$
$$\Rightarrow \nabla \cdot \vec{A}' - \nabla^2 \Lambda + c^{-1}\partial_t \Phi' + c^{-2}\partial_t^2 \Lambda = f$$
$$\Rightarrow \underbrace{\nabla \cdot \vec{A}' + c^{-1}\partial_t \Phi'}_{0,\ \text{by (d)}} + (c^{-2}\partial_t^2 - \nabla^2)\Lambda = f.$$

Hence the required Λ satisfies the well-known inhomogeneous wave equation: $(c^{-2}\partial_t^2 - \nabla^2)\Lambda = f$. This can be solved using standard techniques to yield the required Λ which ensures that the gauge-transformed potentials indeed satisfy the Lorentz condition.

Exercise #45 Take the divergence of the Maxwell equation (163b), to give: $\nabla\cdot\nabla\times\vec{B} - c^{-1}\partial_t \nabla\cdot\vec{E} = 4\pi c^{-1}\nabla\cdot\vec{j}$.
Now, by a standard vector identity, $\nabla\cdot\nabla\times\vec{B} = 0$ and so we have $-c^{-1}\partial_t \nabla\cdot\vec{E} = 4\pi c^{-1}\nabla\cdot\vec{j} \Rightarrow -\partial_t \nabla\cdot\vec{E} = 4\pi\nabla\cdot\vec{j}$. But, from Maxwell equation (163a), $\nabla\cdot\vec{E} = 4\pi\rho$ and so $-\partial_t(\underbrace{4\pi\rho}_{\nabla\cdot\vec{E}}) = 4\pi\nabla\cdot\vec{j}$,
i.e. $\nabla\cdot\vec{j} + \partial_t\rho = 0$. This is the required result. For an interpretation, see any electrodynamics text!

Exercise #46 $\partial_\alpha J^\alpha = 0 \Rightarrow \frac{\partial}{\partial x^0}J^0 + \frac{\partial}{\partial x^1}J^1 + \frac{\partial}{\partial x^2}J^2 + \frac{\partial}{\partial x^3}J^3 = 0$. (a)

Now, $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $J^0 = c\rho$, $J^1 = j^x$, $J^2 = j^y$, $J^3 = j^z$, where (j^x, j^y, j^z) are the cartesian components of the three-current. Hence equation (a) becomes:

$$\frac{\partial}{\partial(ct)}c\rho + \frac{\partial}{\partial x}j^x + \frac{\partial}{\partial y}j^y + \frac{\partial}{\partial z}j^z = 0$$

$$\Rightarrow \partial_t\rho + \nabla\cdot\vec{j} = 0 \quad \text{Q.E.D.}$$

Exercise #47 The Lorentz condition is, from (172), given by:
$\nabla\cdot\vec{A} + c^{-1}\partial_t\Phi = 0$. Let $\vec{A} = (A^x, A^y, A^z)$. Then:

$$\frac{\partial}{\partial x}A^x + \frac{\partial}{\partial y}A^y + \frac{\partial}{\partial z}A^z + \frac{\partial}{\partial(ct)}\Phi = 0. \quad \mathbf{A} = (\Phi, \vec{A}) \Rightarrow \left[\begin{array}{l} A^0 = \Phi \\ A^1 = A^x \\ A^2 = A^y \\ A^3 = A^z \end{array}\right.$$

$$\Rightarrow \frac{\partial}{\partial x^1}A^1 + \frac{\partial}{\partial x^2}A^2 + \frac{\partial}{\partial x^3}A^3 + \frac{\partial}{\partial x^0}A^0 = 0$$

$$\Rightarrow \partial_\alpha A^\alpha = 0 \quad \text{Q.E.D.}$$

Next, we need to show that the following pair of equations:

$$\begin{cases}(\nabla^2 - c^{-2}\partial_t^2)\Phi = -4\pi\rho \\ (\nabla^2 - c^{-2}\partial_t^2)\vec{A} = -4\pi c^{-1}\vec{J}\end{cases} \qquad (a)$$

can be written as $\Box \mathbf{A} = 4\pi c^{-1}\mathbf{J}$, with $\Box \equiv \partial_\alpha \partial^\alpha = \partial_0^2 - \nabla^2$

$= c^{-2}\partial_t^2$

the negative was missing from the notes... sorry!

Now, $\Box \equiv \partial_\alpha \partial^\alpha = \underbrace{g^{\alpha\bullet}\partial_\bullet \partial_\alpha}$

there are 16 terms in this double sum, but only four are nonzero because the metric $g^{\mu\nu}$ is diagonal → see equation (104) of the notes.

$$= g^{00}\partial_0\partial_0 + g^{11}\partial_1\partial_1 + g^{22}\partial_2\partial_2 + g^{33}\partial_3\partial_3$$

$$\overset{(104)}{=} \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3$$

$$\equiv \frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3}$$

$$\overset{(101)}{=} \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)} - \frac{\partial}{\partial x}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\frac{\partial}{\partial z}$$

$$= c^{-2}\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 \equiv c^{-2}\partial_t^2 - \nabla^2.$$

Hence (a) becomes: (b) $\begin{cases}\Box\Phi = 4\pi\rho \\ \Box\vec{A} = 4\pi c^{-1}\vec{J}\end{cases}$

$$\Rightarrow \Box\underbrace{(\Phi, \vec{A})}_{\mathbf{A},\ \text{by } (184)} = (4\pi\rho, 4\pi c^{-1}\vec{J}) = 4\pi c^{-1}\underbrace{(c\rho, \vec{J})}_{\mathbf{J},\ \text{by } (182)}.$$

$$\Rightarrow \Box\mathbf{A} = 4\pi c^{-1}\mathbf{J} \quad \text{Q.E.D.}$$

Exercise #98 (183) ⇒ (a) $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$. Since both α and β can take the values 0, 1, 2, 3, we see that $F^{\alpha\beta}$ is a 4×4 matrix. We see from (a) that, when $\alpha = \beta$, F is zero ⇒ all diagonal elements of F must vanish. What about the other elements? Before working them out, note that: (b) $\partial^0 = g^{0\alpha}\partial_\alpha = \underbrace{g^{00}}_{1}\partial_0 + \underbrace{g^{01}\partial_1 + g^{02}\partial_2 + g^{03}\partial_3}_{\text{all zero since } g^{\mu\nu} \text{ is diagonal}}$

$$= \frac{\partial}{\partial x^0} = \frac{\partial}{\partial(ct)} = \frac{1}{c}\frac{\partial}{\partial t} \equiv c^{-1}\partial_t$$

(c) $\partial^1 = g^{1\alpha}\partial_\alpha = \underbrace{g^{11}}_{-1}\partial_1 = -\partial_1 = -\dfrac{\partial}{\partial x^1} = -\dfrac{\partial}{\partial x} \equiv -\partial_x$ ← note!

(d) Similarly, $\partial^2 = -\partial_y$ & $\partial^3 = -\partial_z$.

(e) From (184), we see that $A^0 = \Phi$, $A^1 = A^x$, $A^2 = A^y$, $A^3 = A^z$, where (A^x, A^y, A^z) are the cartesian components of the vector potential $\vec{A}$.

Now we can work out the remaining components of F:

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = c^{-1}\partial_t A^x - -\partial_x \Phi = c^{-1}\partial_t A^x + \partial_x \Phi \quad (f)$$
$$F^{02} = \partial^0 A^2 - \partial^2 A^0 = c^{-1}\partial_t A^y - -\partial_y \Phi = c^{-1}\partial_t A^y + \partial_y \Phi \quad (g)$$
$$F^{03} = \partial^0 A^3 - \partial^3 A^0 = c^{-1}\partial_t A^z - -\partial_z \Phi = c^{-1}\partial_t A^z + \partial_z \Phi \quad (h)$$

Now, recall equation (166): $\vec{E} = -\nabla\Phi - c^{-1}\partial_t \vec{A}$

$$\Rightarrow -\vec{E} = c^{-1}\partial_t \vec{A} + \nabla\Phi \quad (i)$$

If we write out the components of the vector equation (i), we see that:

$$(j)\quad \begin{cases} -E^x = c^{-1}\partial_t A^x + \partial_x \Phi \\ -E^y = c^{-1}\partial_t A^y + \partial_y \Phi \\ -E^z = \underbrace{c^{-1}\partial_t A^z + \partial_z \Phi} \end{cases}$$

These are the terms on the right sides of (f), (g), (h) respectively.

Bearing (j) in mind, we see that (f)/(g)/(h) become:

(k) $\left\{ F^{01} = -E^x,\ F^{02} = -E^y,\ F^{03} = -E^z \right.$

We have now filled out the top row of equation (190) in the notes. Since $F^{\alpha\beta} = -F^{\beta\alpha}$, the matrix is

~~The first column of (190) is obtained in an analogous manner; I will not bore you with the algebra here!~~

~~What remains is to determine F^{21}, F^{31}, F^{32},~~

antisymmetric and hence we can also write down the first column of (190), by putting in negative signs.

To summarize our progress so far, we have determined:

$$(l) \quad F^{\alpha\beta} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & \times & \times \\ E^y & \times & 0 & \times \\ E^z & \times & \times & 0 \end{pmatrix}$$

where "x" denotes the elements which we still need to find. Let's keep going!

$$(m) \quad \begin{cases} F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\partial_x A^y - -\partial_y A^x = \partial_y A^x - \partial_x A^y \\ F^{13} = \partial^1 A^3 - \partial^3 A^1 = -\partial_x A^z - -\partial_z A^x = \partial_z A^x - \partial_x A^z \\ F^{23} = \partial^2 A^3 - \partial^3 A^2 = -\partial_y A^z - -\partial_z A^y = \partial_z A^y - \partial_y A^z \end{cases}$$

Now, we know from (164) of the notes that $\vec{B} = \nabla \times \vec{A}$. If we write down the three components of this vector equation, we see that:

$$(n) \quad \begin{cases} B^x = \partial_y A^z - \partial_z A^y \\ B^y = \partial_z A^x - \partial_x A^z \\ B^z = \partial_x A^y - \partial_y A^x \end{cases} \Rightarrow (m) \text{ becomes: } (o) \quad \begin{cases} F^{12} = -B^z \\ F^{13} = B^y \\ F^{23} = -B^x \end{cases}$$

Using equation (0), we can fill out some more entries in (1), to give:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & \times & 0 & -B^x \\ E^z & \times & \times & 0 \end{pmatrix}.$$

Finally, make use of the anti-symmetric nature of $F^{\alpha\beta}$ to write down the required result, namely equation (190) of the notes.

<u>Exercise #49</u> $\partial_\alpha F^{\alpha\beta} = 4\pi c^{-1} J^\beta$ (a)

There are four equations here, with cases $\beta = 0,1,2,3$.

$$\Rightarrow \partial_0 F^{0\beta} + \partial_1 F^{1\beta} + \partial_2 F^{2\beta} + \partial_3 F^{3\beta} = 4\pi c^{-1} J^\beta \quad \text{(b)}$$

let $\beta = 0$ in (b)

$$\Rightarrow \partial_0 \underbrace{F^{00}}_{0} + \partial_1 \underbrace{F^{10}}_{E^x} + \partial_2 \underbrace{F^{20}}_{E^y} + \partial_3 \underbrace{F^{30}}_{E^z} = 4\pi c^{-1} \underbrace{J^0}_{c\rho}$$

$$\Rightarrow \frac{\partial}{\partial x^1} E^x + \frac{\partial}{\partial x^2} E^y + \frac{\partial}{\partial x^3} E^z = 4\pi c^{-1} c \rho$$

$$\Rightarrow \frac{\partial}{\partial x} E^x + \frac{\partial}{\partial y} E^y + \frac{\partial}{\partial z} E^z = 4\pi \rho$$

$$\Rightarrow \underline{\underline{\nabla \cdot \vec{E} = 4\pi\rho}}$$ which is Maxwell equation (163a).

let $\beta = 1$ in (b) $\Rightarrow \partial_0 \underbrace{F^{01}}_{-E^x} + \partial_1 \underbrace{F^{11}}_{0} + \partial_2 \underbrace{F^{21}}_{B^z} + \partial_3 \underbrace{F^{31}}_{-B^y} = 4\pi c^{-1} \underbrace{J^1}_{j^x}$ 26/5/2

$$\Rightarrow \frac{\partial}{\partial x^0}(-E^x) + \frac{\partial}{\partial x^2} B^z + \frac{\partial}{\partial x^3}(-B^y) = 4\pi c^{-1} j^x$$

$$\Rightarrow \frac{-\partial}{\partial (ct)} E^x + \frac{\partial}{\partial y} B^z - \frac{\partial}{\partial z} B^y = 4\pi c^{-1} j^x$$

$$\Rightarrow -c^{-1}\partial_t E^x + \partial_y B^z - \partial_z B^y = 4\pi c^{-1} j^y$$

This is equal to the x-component of the other inhomogeneous Maxwell equation, (163b):

$$\underline{\underline{\nabla \times \vec{B} - c^{-1}\partial_t \vec{E} = 4\pi c^{-1}\vec{j}}}$$

The y and z components of this maxwell equation follow from the $\beta = 2$, $\beta = 3$ cases of (b) using similar logic.

Next, we need to look at the equation:

(c) $\partial^\alpha F^{\beta\gamma} + \partial^\gamma F^{\alpha\beta} + \partial^\beta F^{\gamma\alpha} = 0$

because we have 3 free spacetime indices

There are, on the surface, $4^3 = 64$ equations here! We need to narrow this list down! When $\alpha = \beta = \gamma$ we get $0 = 0$; also, if any two the α, β, γ are equal then (c) reduces to $0 = 0$, because $F^{\alpha\beta} = -F^{\beta\alpha}$. Hence we need only consider the cases $\alpha \neq \beta \neq \gamma$. Now, since (c) is invariant under any permutation of the indices α, β, γ, we can therefore only consider $\alpha < \beta < \gamma$. This leaves us with only 4 cases, shown in the box on the left. We work out each of those in turn.

The four nontrivial distinct cases of the 64 equations in (c).

α	β	γ
0	1	2
0	1	3
0	2	3
1	2	3

$\underline{\underline{\alpha = 0, \beta = 1, \gamma = 2}}$

$$\underbrace{\partial^0}_{c^{-1}\partial_t \text{ (see p. 42)}} \underbrace{F^{12}}_{-B^z \text{ (see p. 43)}} + \overbrace{\partial^2}^{-\partial_y \text{ (see p. 42)}} \underbrace{F^{01}}_{-E^x \text{ (see top p. 43)}} + \overbrace{\partial^1}^{-\partial_x \text{ (see p. 42)}} \underbrace{F^{20}}_{E^y \text{ (see top p. 43)}} = 0$$

$$\Rightarrow \quad c^{-1}\partial_t(-B^z) - \partial_y(-E^x) - \partial_x(E^y) = 0$$

$$\Rightarrow \quad \text{(d)} \quad c^{-1}\partial_t B^z + \partial_x E^y - \partial_y E^x = 0$$

$\underline{\underline{\alpha = 0, \beta = 1, \gamma = 3}}$

$$\partial^0 F^{13} + \partial^3 F^{01} + \partial^1 F^{30} = 0$$

$$c^{-1}\partial_t B^y - \partial_z(-E^x) - \partial_x(E^z) = 0$$

$$\Rightarrow \quad \text{(e)} \quad c^{-1}\partial_t B^y + \partial_z E^x - \partial_x E^z = 0$$

$\underline{\alpha = 0, \beta = 2, \gamma = 3}$ $\quad \partial^0 F^{23} + \partial^3 F^{02} + \partial^2 F^{30} = 0$

$$c^{-1}\partial_t(-B^x) - \partial_z(-E^y) - \partial_y(E^z) = 0$$

$$\Rightarrow (f) \quad c^{-1}\partial_t B^x + \partial_y E^z - \partial_z E^y = 0$$

Equations (d), (e), (f) are respectively equal to the z, y, x components of the maxwell equation:

$$\underline{c^{-1}\partial_t \vec{B} + \nabla \times \vec{E} = 0}$$

$\underline{\alpha = 1, \beta = 2, \gamma = 3}$ Therefore (c) becomes: $\partial^1 F^{23} + \partial^3 F^{12} + \partial^2 F^{31} = 0$

$$\Rightarrow -\partial_x(-B^x) - \partial_z(-B^z) - \partial_y(-B^y) = 0$$

$$\Rightarrow \underline{\nabla \cdot \vec{B} = 0}$$

... which is the last remaining maxwell equation.

Exercise #50 Start with (192): (a) $\dfrac{dp^\alpha}{dt} = \dfrac{q}{c} F^{\alpha\beta} u_\beta$.

We will separately consider the cases $\alpha = 0, 1, 2, 3$ of the four equations in (a).

$\underline{\alpha = 0}$ $\Rightarrow$ (a) becomes $\dfrac{dp^0}{dt} = \dfrac{q}{c} F^{0\beta} u_\beta = \dfrac{q}{c}\left(\underbrace{F^{00}}_{0} u_0 + F^{01} u_1 + F^{02} u_2 + F^{03} u_3\right)$

$F^{01} = -E^x$
$F^{02} = -E^y$
$F^{03} = -E^z$
see top p. 43

$$\Rightarrow \frac{dp^0}{dt} = -\frac{q}{c}\left(E^x u_1 + E^y u_2 + E^z u_3\right)$$

$$= -\frac{q}{c} \vec{E} \cdot \vec{u}$$ ... which is identical to (178). Q.E.D.

p^x = spatial x component of three-momentum

$\underline{\alpha = 1}$ $\Rightarrow$ (a) becomes $\dfrac{dp^1}{dt} = \dfrac{q}{c} F^{1\beta} u_\beta = \dfrac{q}{c}\left(\underbrace{F^{10}}_{E^x} u_0 + \underbrace{F^{11}}_{0} u_1 + \underbrace{F^{12}}_{-B^z} u_2 + \underbrace{F^{13}}_{B^y} u_3\right)$

$$\Rightarrow \frac{dp^x}{dt} = \frac{q}{c}\left(u_0 E^x - B^z u_2 + B^y u_3\right) \quad (b)$$

Now, $u_0 = u^0$, and $u_2 = -u^2$ and $u_3 = -u^3$. Hence (b) becomes:

(c) $$\frac{dp^x}{d\tau} = \frac{q}{c}\left(u^0 E^x + u^2 B^z - u^3 B^y\right)$$

Equation (c) is the x-component of equation (176) of the notes. The other two components of (176) follow from the $\alpha = 2, 3$ cases of (a) using a similar argument.

Exercise #51 Before beginning the question, note from exercise #28 that:

(a) $p^{0'}_0 = \gamma,\ p^{0'}_1 = -\frac{\gamma v}{c},\ p^{1'}_0 = -\frac{\gamma v}{c},\ p^{1'}_1 = \gamma,\ p^{2'}_2 = 1,\ p^{3'}_3 = 1,\ p^{0'}_2 = p^{0'}_3 = p^{1'}_2 = p^{1'}_3 = p^{2'}_0 = p^{2'}_1 = p^{2'}_3 = p^{3'}_0 = p^{3'}_1 = p^{3'}_2 = 0$

(You may want to have a brief look at the solutions to exercise #28 before looking at what is written below.)

$\boxed{E^x}$ $$\underbrace{F^{1'0'}}_{\text{see top p. 43}} = E^{x'} = \underbrace{p^{1'}_\bullet p^{0'}_\blacksquare}\, F^{\bullet\blacksquare}$$

because $F^{\mu\nu}$ is a tensor → we only keep the terms where both p's are nonzero, and the F component is zero... all the remaining terms are crossed out

$$\begin{aligned}\Rightarrow E^{x'} = {} & \cancel{p^{1'}_0 p^{0'}_0 F^{00}} + p^{1'}_0 p^{0'}_1 F^{01} + \cancel{p^{1'}_0 p^{0'}_2 F^{02}} + \cancel{p^{1'}_0 p^{0'}_3 F^{03}} \\ & + p^{1'}_1 p^{0'}_0 F^{10} + \cancel{p^{1'}_1 p^{0'}_1 F^{11}} + \cancel{p^{1'}_1 p^{0'}_2 F^{12}} + \cancel{p^{1'}_1 p^{0'}_3 F^{13}} \\ & + \cancel{p^{1'}_2 p^{0'}_0 F^{20}} + \cancel{p^{1'}_2 p^{0'}_1 F^{21}} + \cancel{p^{1'}_2 p^{0'}_2 F^{22}} + \cancel{p^{1'}_2 p^{0'}_3 F^{23}} \\ & + \cancel{p^{1'}_3 p^{0'}_0 F^{30}} + \cancel{p^{1'}_3 p^{0'}_1 F^{31}} + \cancel{p^{1'}_3 p^{0'}_2 F^{32}} + \cancel{p^{1'}_3 p^{0'}_3 F^{33}}\end{aligned}$$

row crossed out since $p^{1'}_2 = 0$

row crossed out since $p^{1'}_3 = 0$

column crossed out since $p^{0'}_2 = 0$

column crossed out since $p^{0'}_3 = 0$

Diagonal crossed out since $F^{00} = F^{11} = F^{22} = F^{33} = 0$

$$\Rightarrow E^{x'} = p^{1'}_0 p^{0'}_1 F^{01} + p^{1'}_1 p^{0'}_0 F^{10} = \left(-\frac{\gamma v}{c}\right)\left(-\frac{\gamma v}{c}\right)\left(-E^x\right) + (\gamma)(\gamma)(E^x)$$
$$= E^x \gamma^2\left(-v^2/c^2 + 1\right) = E^x \gamma^2 \gamma^{-2} = E^x \quad \text{Q.E.D.}$$

$\boxed{E^y}$ $$F^{2'0'} = E^{y'} = p^{2'}_\bullet p^{0'}_\blacksquare F^{\bullet\blacksquare} = \underbrace{p^{2'}_2 p^{0'}_0 F^{20} + p^{2'}_2 p^{0'}_1 F^{21}} = (1)(\gamma)(E^y) + (1)\left(-\frac{\gamma v}{c}\right)(B^z)$$
$$= \gamma\left(E^y - vB^z/c\right) \quad \text{Q.E.D.}$$

I have only written down the nonzero terms!!

$\boxed{E^z}$ $$F^{3'0'} = E^{z'} = p^{3'}_\bullet p^{0'}_\blacksquare F^{\bullet\blacksquare} = p^{3'}_3 p^{0'}_0 F^{30} + p^{3'}_3 p^{0'}_1 F^{31} = (1)(\gamma)(E^z) + (1)\left(-\frac{\gamma v}{c}\right)(-B^y)$$
$$= \gamma\left(E^z + vB^y/c\right) \quad \text{Q.E.D.}$$

This completes the derivation of (194a). The derivation of (194b) uses similar logic, and will not be given here!

Exercise #52 The v-reversed form of equations (194) is obtained by changing v to $-v$, and interchanging primed & unprimed quantities.

Thus: (a) $$\begin{cases} E^x = E^{x'},\ E^y = \gamma(E^{y'} + vB^{z'}/c),\ E^z = \gamma(E^{z'} - vB^{y'}/c) \\ B^x = B^{x'},\ B^y = \gamma(B^{y'} - vE^{z'}/c),\ B^z = \gamma(B^{z'} + vE^{y'}/c)\end{cases}$$

We can see from the symmetry of Figure 25 that the electric field, in either frame, has no components along the $x^3 \equiv z$ axis. Thus $E^z = E^{z'} = 0$ and so (a) becomes:

(b) $\left[\begin{array}{l} E^x = E^{x'},\ E^y = \gamma(E^{y'} + VB^{z'}/c),\ 0 = B^{y'} \\ B^x = B^{x'},\ B^y = \gamma B^{y'},\ B^z = \gamma(B^{z'} + \frac{V}{c}E^{y'}) \end{array}\right.$ $B^{y'} = B^y = 0$

(c) $[E^x = E^{x'}, E^y = \gamma(E^{y'} + VB^{z'}/c, B^x = B^{x'}, B^z = \gamma(B^{z'} + VE^{y'}/c)$, other components vanish

$\Rightarrow E^x = E^{x'} \stackrel{(196)}{=} \dfrac{-q\gamma Vt}{(b^2+\gamma^2V^2t^2)^{3/2}}$ Q.E.D.

$B^{y'} = B^{z'}$

To proceed further, we need to put $B^{x'} = 0$ in (c), since there is no magnetic field in frame S'. Thus (c) becomes:

(d) $[E^x = E^{x'}, E^y = \gamma E^{y'}, B^z = \gamma VE^{y'}/c$, other components vanish.

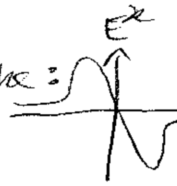
$\Rightarrow E^y = \gamma E^{y'} \stackrel{(196)}{=} \dfrac{\gamma q b}{(b^2+\gamma^2V^2t^2)^{3/2}}$ Q.E.D.

$\Rightarrow B^z = \underbrace{\gamma V}E^{y'}/c$ Q.E.D.

$\rightarrow$ sorry for the typo in the notes!

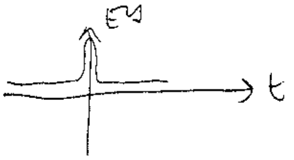
Exercise #53 The E^x component given by the first member of (197) is $E^x = -\dfrac{qV\gamma t}{(b^2+\gamma^2V^2t^2)^{3/2}}$. A graph of E^x versus t

looks like:



By symmetry, this integrates to zero. As γ gets larger, the graph of E^x versus time becomes progressively more sharply peaked. When γ is too large, the detector — which effectively integrates over its smallest resolvable timescale — will read zero. Hence E^x becomes irrelevant for any realistic detector in the limit of high velocity.

Thus, for a realistic detector and a very fast particle, (197) becomes: $E_y = \dfrac{\gamma q b}{(b^2+\gamma^2V^2t^2)^{3/2}}$, $B_z = VE^y/c$. When γ is very large, E^y and B^z are negligible for $t \neq 0$ and very highly peaked at $t = 0$.



Thus the observer at P sees a pulse, at $t = 0$, of perpendicular electric field E^y and magnetic field B^z.

Exercise #54

$$\vec{E} = (E^x, E^y, \overset{0}{\overset{\shortparallel}{E^z}}) \overset{(197)}{=} \left(\frac{-qv\gamma t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}, \frac{\gamma q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}, 0 \right)$$

$$= \frac{\gamma q}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \underbrace{(-vt, b, 0)}_{r\vec{n} \text{ (see figure 25)}}$$

let $\vec{r} \equiv r\vec{n}$

$$= \frac{\gamma q\, r\vec{n}}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$= \frac{\gamma q \vec{r}}{(b^2+\gamma^2 v^2 t^2)^{3/2}} = \frac{q\vec{r}}{\frac{\gamma^2}{\gamma^3}(b^2+\gamma^2 v^2 t^2)^{3/2}} = \frac{q\vec{r}}{\gamma^2\left(\frac{b^2}{\gamma^2}+v^2t^2\right)^{3/2}} \quad (a)$$

Now, $\frac{1}{\gamma^2} = \left(\frac{1}{\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)}\right)^2 = 1-\frac{v^2}{c^2} \equiv 1-\beta^2$, with $\beta \equiv \frac{v}{c}$.

Hence $\left(\frac{b^2}{\gamma^2}+v^2t^2\right)^{3/2} = \left(b^2(1-\beta^2)+v^2t^2\right)^{3/2} = \left(\underbrace{b^2+v^2t^2}_{r^2}-b^2\beta^2\right)^{3/2} = (r^2-b^2\beta^2)^{3/2}$

$$= \left(r^2\left(1-\frac{b^2\beta^2}{r^2}\right)\right)^{3/2}$$

$$= r^3\left(1-\left(\frac{b\beta}{r}\right)^2\right)^{3/2} \quad (b)$$

Now, $\sin\psi = \sin\theta = b/r$. Thus (b) becomes:

(c) $\left(\frac{b^2}{\gamma^2}+v^2t^2\right)^{3/2} = r^3(1-\beta^2\sin^2\psi)^{3/2}$. Substitute (c) into (a) to give: $\vec{E} = \frac{q\vec{r}}{\gamma^2 r^3(1-\beta^2\sin^2\psi)^{3/2}}$, which is the required result.

Exercise #55 $\mathcal{L} = \frac{1}{2}(\phi_{,\alpha}\phi_{,}{}^{\alpha} - \mu^2\phi^2) \equiv \frac{1}{2}[(\partial_\blacksquare\phi)(\partial^\blacksquare\phi) - \mu^2\phi^2]$.

Now, the Euler Lagrange equation is: $\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right) = 0$, which may also be written as: $\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right) = 0$. This becomes, for our Lagrangian density:

$$\frac{\partial}{\partial\phi}\left(-\frac{1}{2}\mu^2\phi^2\right) - \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\frac{1}{2}(\partial_\blacksquare\phi)(\partial^\blacksquare\phi)\right] = 0$$

$$-\mu^2\phi - \frac{1}{2}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha\phi)}\left[(\partial_\blacksquare\phi)g^{\blacksquare\bullet}(\partial_\bullet\phi)\right] = 0$$

can be pulled out the front of the differential operators.

$$-\mu^2\phi - \frac{1}{2}g^{\blacksquare\bullet}\frac{\partial}{\partial x^\alpha}\underbrace{\frac{\partial}{\partial(\partial_\alpha\phi)}\left[(\partial_\blacksquare\phi)(\partial_\bullet\phi)\right]} = 0$$

we now use the product rule.

$$-\mu^2\phi - \frac{1}{2} g^{\blacksquare\bullet} \underbrace{\frac{\partial}{\partial x^\alpha}}_{\equiv \partial_\alpha}\left[(\partial_\blacksquare\phi)\underbrace{\frac{\partial}{\partial(\partial_\alpha\phi)}(\partial_\bullet\phi)}_{\delta^\alpha_\bullet} + (\partial_\bullet\phi)\underbrace{\frac{\partial}{\partial(\partial_\alpha\phi)}(\partial_\blacksquare\phi)}_{\delta^\alpha_\blacksquare}\right] = 0$$

$$-\mu^2\phi - \frac{1}{2} g^{\blacksquare\bullet}\partial_\alpha\left[(\partial_\blacksquare\phi)\delta^\alpha_\bullet + (\partial_\bullet\phi)\delta^\alpha_\blacksquare\right] = 0$$

$$-\mu^2\phi - \frac{1}{2} g^{\blacksquare\bullet}\underbrace{\delta^\alpha_\bullet\partial_\alpha}_{\partial_\bullet}\partial_\blacksquare\phi - \frac{1}{2} g^{\blacksquare\bullet}\underbrace{\delta^\alpha_\blacksquare\partial_\alpha}_{\partial_\blacksquare}\partial_\bullet\phi = 0$$

$$-\mu^2\phi - \frac{1}{2}\underbrace{g^{\blacksquare\bullet}\partial_\bullet}_{\partial^\blacksquare}\partial_\blacksquare\phi - \frac{1}{2}\underbrace{g^{\blacksquare\bullet}\partial_\blacksquare}_{\partial^\bullet}\partial_\bullet\phi = 0$$

$$-\mu^2\phi - \frac{1}{2}\underbrace{\partial^\blacksquare\partial_\blacksquare}_{\Box}\phi - \frac{1}{2}\underbrace{\partial^\bullet\partial_\bullet}_{\Box}\phi = 0$$

$$-\mu^2\phi - \frac{1}{2}\Box\phi - \frac{1}{2}\Box\phi = 0 \Rightarrow (\Box + \mu^2)\phi = 0 \quad \text{Q.E.D.}$$

Exercise #56 $\mathcal{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha$ (a). Now, we know from (188) that: (b) $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. It can be shown, and I will give the proof at the end of the exercise, that (b) implies (c) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Given (b) and (c), the Lagrangian density in (a) becomes: (d) $\mathcal{L} = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{c}J_\mu A^\mu$. We consider each component of the four vector A^β to be an independent field, and so the Euler-Lagrange equations become: (e) $\dfrac{\partial\mathcal{L}}{\partial A^\beta} - \dfrac{\partial}{\partial x^\alpha}\left(\dfrac{\partial\mathcal{L}}{\partial A^\beta{}_{,\alpha}}\right) = 0$,

i.e.: (f) $\dfrac{\partial\mathcal{L}}{\partial A^\beta} - \dfrac{\partial}{\partial x^\alpha}\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\beta)}\right) = 0$. Now substitute (d) into (f):

$$\Rightarrow 0 = \frac{\partial}{\partial A^\beta}\left(\left(-\frac{1}{c}J_\mu A^\mu\right)\right) - \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha A^\beta)}\left[\left(-\frac{1}{16\pi}\right)(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\right]$$

(g) $$= -c^{-1}J_\beta + \frac{1}{16\pi}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha A^\beta)}\Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - (\partial_\nu A^\mu)(\partial^\mu A^\nu) + (\partial_\nu A_\mu)(\partial^\nu A^\mu)\Big]$$

The first and fourth terms in the square brackets are equal (remember that μ,ν are dummy indices!). Similarly, the second and third terms are equal.

Hence:

$$\text{(h)}\quad 0 = -c^{-1}J_\beta + \frac{\overset{\text{note!}}{2}}{16\pi}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha A^\beta)}\Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\Big]$$

Now, since we are differentiating with respect to $\partial_\alpha A^\beta$, each term in the square brackets had better have a lower index for the "∂" and an upper index for the "A". Hence we need to do this:

$$\text{(i)}\quad \left[\begin{array}{ll} \partial_\mu A_\nu = g_{\bullet\nu}\,\partial_\mu A^\bullet, & \partial_\mu A_\nu = g_{\bullet\nu}\,\partial_\mu A^\bullet \\ \partial^\mu A^\nu = g^{\blacksquare\mu}\,\partial_\blacksquare A^\nu, & \partial^\nu A^\mu = g^{\blacksquare\nu}\,\partial_\blacksquare A^\mu \end{array}\right.$$

Given (i), (h) becomes:

$$0 = -c^{-1}J_\beta + \frac{1}{8\pi}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha A^\beta)}\Big[g_{\bullet\nu}(\partial_\mu A^\bullet)g^{\blacksquare\mu}(\partial_\blacksquare A^\nu) - g_{\bullet\nu}(\partial_\mu A^\bullet)g^{\blacksquare\nu}(\partial_\blacksquare A^\mu)\Big]$$

$$= -c^{-1}J_\beta + \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\mu}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha A^\beta)}\Big[(\partial_\mu A^\bullet)(\partial_\blacksquare A^\nu)\Big]$$

$$- \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\nu}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial(\partial_\alpha A^\beta)}\Big[(\partial_\mu A^\bullet)(\partial_\blacksquare A^\mu)\Big]$$

now we use the product rule

and the fact that:

$$\frac{\partial}{\partial(\partial_\alpha A^\beta)}(\partial_\gamma A^\delta) = \delta^\alpha_\gamma\,\delta^\delta_\beta$$

$$= -c^{-1}J_\beta + \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\mu}\frac{\partial}{\partial x^\alpha}\Big[(\partial_\mu A^\bullet)\delta^\alpha_\blacksquare\delta^\nu_\beta + (\partial_\blacksquare A^\nu)\delta^\alpha_\mu\delta^\bullet_\beta\Big]$$

$$- \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\nu}\frac{\partial}{\partial x^\alpha}\Big[(\partial_\mu A^\bullet)\delta^\alpha_\blacksquare\delta^\mu_\beta + (\partial_\blacksquare A^\mu)\delta^\alpha_\mu\delta^\bullet_\beta\Big]$$

$$= -c^{-1}J_\beta + \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\mu}\delta^\alpha_\blacksquare\delta^\nu_\beta\,\partial_\alpha\partial_\mu A^\bullet$$

$$+ \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\mu}\delta^\alpha_\mu\delta^\bullet_\beta\,\partial_\alpha\partial_\blacksquare A^\nu$$

$$- \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\nu}\delta^\alpha_\blacksquare\delta^\mu_\beta\,\partial_\alpha\partial_\mu A^\bullet - \frac{1}{8\pi}g_{\bullet\nu}g^{\blacksquare\nu}\delta^\alpha_\mu\delta^\bullet_\beta\,\partial_\alpha\partial_\blacksquare A^\mu$$

(now apply the Kronecker δ's to the g's, where this can be done)

$$= -c^{-1}J_\beta + \frac{1}{8\pi}g_{\bullet\beta}g^{\alpha\mu}\partial_\alpha\partial_\mu A^\bullet + \frac{1}{8\pi}g_{\beta\nu}g^{\blacksquare\alpha}\partial_\alpha\partial_\blacksquare A^\nu$$

$$- \frac{1}{8\pi}g_{\bullet\nu}g^{\alpha\nu}\delta^\mu_\beta\,\partial_\alpha\partial_\mu A^\bullet - \frac{1}{8\pi}g_{\beta\nu}g^{\blacksquare\nu}\delta^\alpha_\mu\,\partial_\alpha\partial_\blacksquare A^\mu$$

~~(now apply the g's to the ∂'s where possible)~~

$$\cancel{= -c^{-1}J_\beta + \frac{1}{8\pi}}$$

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$$\Rightarrow 8\pi c^{-1} J_\beta = \partial_\alpha \Big[g_{\bullet\beta} \underbrace{g^{\alpha\mu}\partial_\mu}_{\to\,\partial^\alpha} A^\bullet + g_{\beta\nu} \underbrace{g^{\blacksquare\alpha}\partial_\blacksquare}_{\to\,\partial^\alpha} A^\nu - \underbrace{g_{\bullet\nu} g^{\alpha\nu}}_{\to\,\delta^\alpha_\bullet} \delta^\mu_\beta \partial_\mu A^\bullet - \underbrace{g_{\beta\nu} g^{\blacksquare\nu}}_{\to\,\delta^\blacksquare_\beta} \delta^\alpha_\mu \partial_\blacksquare A^\mu \Big]$$

see equation (A9) of Rindler's appendix, and remember that $g^{\mu\nu} = g^{\nu\mu}$.

$$8\pi c^{-1} J_\beta = \partial_\alpha \Big[g_{\bullet\beta}\partial^\alpha A^\bullet + g_{\beta\nu}\partial^\alpha A^\nu - \delta^\alpha_\bullet \delta^\mu_\beta \partial_\mu A^\bullet - \delta^\blacksquare_\beta \delta^\alpha_\mu \partial_\blacksquare A^\mu \Big]$$

$$= \partial_\alpha \left[\partial^\alpha A_\beta + \partial^\alpha A_\beta - \partial_\beta A^\alpha - \partial_\beta A^\alpha \right] = 2\partial_\alpha \left[\partial^\alpha A_\beta - \partial_\beta A^\alpha \right]$$

$$\Rightarrow 4\pi c^{-1} J_\beta = \partial_\alpha \partial^\alpha A_\beta - \partial_\alpha \partial_\beta A^\alpha$$

~~We now need to raise the subscript in the J... apply $g_{\gamma\beta}$ to both sides...~~

$$\cancel{\Rightarrow 4\pi c^{-1} g_{\gamma\beta} J^\gamma = g_{\gamma\beta}\partial_\alpha\partial^\alpha A_\beta - g_{\gamma\beta}\partial_\alpha\partial_\beta A^\alpha}$$

~~Now apply $g^{\delta\beta}$ to both sides...~~

$$\cancel{\Rightarrow 4\pi c^{-1} \underbrace{g^{\delta\beta} g_{\gamma\beta}}_{\to\,\delta^\delta_\gamma} J^\gamma = g^{\delta\beta} g_{\gamma\beta}\partial_\alpha\partial^\alpha A_\beta - g^{\delta\beta} g_{\gamma\beta}\partial_\alpha\partial_\beta A^\alpha}$$

$$\cancel{4\pi c^{-1} \delta^\delta_\gamma J^\gamma = \delta^\delta_\gamma \partial_\alpha\partial^\alpha A_\beta - \delta^\delta_\gamma \partial_\alpha\partial_\beta A^\alpha}$$

$$\cancel{4\pi c^{-1} J^\delta =}$$

We now need to raise the subscript on the J_β. Therefore, apply $g^{\bullet\beta}$ to both sides:

$$\Rightarrow 4\pi c^{-1} g^{\bullet\beta} J_\beta = g^{\bullet\beta}\partial_\alpha\partial^\alpha A_\beta - g^{\bullet\beta}\partial_\alpha\partial_\beta A^\alpha$$

$$4\pi c^{-1} J^\bullet = \partial_\alpha\partial^\alpha A^\bullet - \partial_\alpha\partial^\bullet A^\alpha$$

$$4\pi c^{-1} J^\bullet = \partial_\alpha(\partial^\alpha A^\bullet - \partial^\bullet A^\alpha) = \partial_\alpha F^{\alpha\bullet}$$

Let $\bullet = \beta$ (it's a free index)

$$\Rightarrow \underline{\underline{4\pi c^{-1} J^\beta = \partial_\alpha F^{\alpha\beta}}} \quad \text{Q.E.D.}$$

$$4\pi c^{-1} J^\beta = \partial_\alpha F^{\alpha\beta}$$

$$\Rightarrow 4\pi c^{-1}\partial_\beta J^\beta = \boxed{\partial_\alpha\partial_\beta F^{\alpha\beta}} = 0$$

this is zero, because $F^{\alpha\beta}$ is antisymmetric and $\partial_\alpha\partial_\beta$ is symmetric

$$\Rightarrow \underline{\underline{\partial_\beta J^\beta = 0}} \quad \text{Q.E.D.}$$

Addendum We end with the promised proof that (b) leads to (c).

$$(b) \Rightarrow F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

$$\Rightarrow g_{\bullet\mu} g_{\blacksquare\nu} F^{\mu\nu} = g_{\bullet\mu} g_{\blacksquare\nu} \partial^{\mu} A^{\nu} - g_{\bullet\mu} g_{\blacksquare\nu} \partial^{\nu} A^{\mu}$$

$$\Rightarrow F_{\bullet\blacksquare} = \partial_{\bullet} A_{\blacksquare} - \partial_{\blacksquare} A_{\bullet}$$

$$\Rightarrow F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \quad \text{Q.E.D.}$$