Solutions to Honours Quantum Mechanics Exercises

$\boxed{1-18}$

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Exercise #1 $\Psi^{\mp} \equiv e^{\pm imt}$. Begin by showing that these are indeed solutions of the Klein-Gordon equation:

$$
\begin{aligned}
(\partial_t^2 - \nabla^2 + m^2)\Psi^{\mp} &= (\partial_t^2 - \nabla^2 + m^2)e^{\pm imt} \\
&= ((\pm im)^2 - \nabla^2 + m^2)e^{\pm imt} \\
&= (-m^2 - \nabla^2 + m^2)\Psi^{\mp} \\
&= -\nabla^2 \Psi^{\mp} \\
&= 0 \quad \text{Q.E.D.}
\end{aligned}
$$

To show that the $\Psi^{\mp}$ are solutions of negative and positive energy, we apply the energy operator:

$$
\begin{aligned}
E\Psi^{\mp} = i\partial_t \Psi^{\mp} &= i\partial_t e^{\pm imt} \\
&= i(\pm im)e^{\pm imt} \\
&= \mp m e^{\pm imt} \\
&= \mp \Psi^{\mp} \quad \text{Q.E.D.}
\end{aligned}
$$

These are particles at rest (zero momentum) which have opposite signs for their rest energy.

Exercise #2

$$
\begin{aligned}
\nabla\cdot(A\nabla B) &\equiv (\partial_x, \partial_y, \partial_z)\cdot(A(\partial_x B, \partial_y B, \partial_z B)) \\
&= (\partial_x, \partial_y, \partial_z)\cdot(A\partial_x B, A\partial_y B, A\partial_z B) \\
&= \partial_x A\partial_x B + \partial_y A\partial_y B + \partial_z A\partial_z B \\
&= \partial_x(A\partial_x B) + \partial_y(A\partial_y B) + \partial_z(A\partial_z B)
\end{aligned}
$$

(now use product rule)

$$
\begin{aligned}
&= A\partial_x^2 B + (\partial_x A)(\partial_x B) + A\partial_y^2 B + (\partial_y A)(\partial_y B) \\
&\quad + A\partial_z^2 B + (\partial_z A)(\partial_z B) \\
&= A(\partial_x^2 B + \partial_y^2 B + \partial_z^2 B) + (\partial_x A, \partial_y A, \partial_z A)\cdot \\
&\qquad (\partial_x B, \partial_y B, \partial_z B)
\end{aligned}
$$

$= A\nabla^2 B + \nabla A \cdot \nabla B$ Q.E.D.

Next, we show how to get from equation (20) to equation (21) of the lecture notes.

$$(20)\quad \underbrace{\Phi^* \partial_t^2 \Phi - \Phi \partial_t^2 \Phi^*}_{} + \underbrace{\Phi \nabla^2 \Phi^*}_{} - \underbrace{\Phi^* \nabla^2 \Phi}_{} = 0$$

This can be written as $\partial_t(\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*)$, as is easily verified.

From the formula which we have just proven,

$$\nabla \cdot (\Phi \nabla \Phi^*) = \nabla\Phi \cdot \nabla\Phi^* + \Phi \nabla^2 \Phi^*$$

$$\Rightarrow \Phi \nabla^2 \Phi^* = \nabla \cdot (\Phi \nabla \Phi^*) - \nabla\Phi \cdot \nabla\Phi^*$$

Similarly, we have:

$$\Phi^* \nabla^2 \Phi = \nabla \cdot (\Phi^* \nabla \Phi) - \nabla\Phi^* \cdot \nabla\Phi$$

Therefore (20) can be re-written as:

$$\partial_t(\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*) + \nabla \cdot (\Phi \nabla \Phi^*) - \cancel{\nabla\Phi \cdot \nabla\Phi^*} - \nabla \cdot (\Phi^* \nabla \Phi) + \cancel{\nabla\Phi^* \cdot \nabla\Phi} = 0$$

$$\partial_t(\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*) + \nabla \cdot (\Phi \nabla \Phi^* - \Phi^* \nabla \Phi) = 0$$

Q.E.D.

We perform these manipulations so as to write (20) in a form which may be directly compared to the continuity equation (17). This is an essential step in our derivation of expressions for the Klein-Gordon current and probability density.

<u>Exercise #3</u> There is no assumption involved in writing down the ansatz (25).

Substitute (25) into (23a) to give:

$$\rho = \Theta(\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*)$$

$$= \Theta(\tilde{\Phi}^* e^{+imt} \partial_t \tilde{\Phi} e^{-imt} - \tilde{\Phi} e^{-imt} \partial_t \tilde{\Phi}^* e^{imt})$$

$= \Theta\big(\tilde{\Phi}^* \cancel{e^{imt}} \tilde{\Phi}(-im \cancel{e^{-imt}}) + \tilde{\Phi}^* \cancel{e^{imt}} (\partial_t \tilde{\Phi}) \cancel{e^{-imt}}$
$- \tilde{\Phi} \cancel{e^{-imt}} \tilde{\Phi}^* (im \cancel{e^{imt}}) - \tilde{\Phi} \cancel{e^{-imt}} (\partial_t \tilde{\Phi}^*) \cancel{e^{imt}}\big)$

$= \Theta(-im|\tilde{\Phi}|^2 + \tilde{\Phi}^* \partial_t \tilde{\Phi} - im|\tilde{\Phi}|^2 - \tilde{\Phi}\partial_t \tilde{\Phi}^*)$

$= \Theta(-2im|\tilde{\Phi}|^2 + \tilde{\Phi}^* \partial_t \tilde{\Phi} - \tilde{\Phi}\partial_t \tilde{\Phi}^*)$ Q.E.D.

This is useful in the context of our search for Θ, because the nonrelativistic limit of this expression: $\rho \to -2im\Theta|\tilde{\Phi}|^2 = -2im\Theta|\Phi|^2$, must be equal to $|\Phi|^2$, thus allowing us to "fix" Θ.

<u>Exercise #4</u> $\sigma_x^2 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1$

$\sigma_y^2 = \begin{pmatrix}0&-i\\+i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1$

$\sigma_z^2 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1$

$\sigma_x\sigma_y = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}i&0\\0&-i\end{pmatrix} = i\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i\sigma_z$

$-\sigma_y\sigma_x = -\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = -\begin{pmatrix}-i&0\\0&i\end{pmatrix} = i\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i\sigma_z$

$\sigma_y\sigma_z = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&i\\i&0\end{pmatrix} = i\begin{pmatrix}0&1\\1&0\end{pmatrix} = i\sigma_x$

$-\sigma_z\sigma_y = -\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = -\begin{pmatrix}0&-i\\-i&0\end{pmatrix} = i\begin{pmatrix}0&1\\1&0\end{pmatrix} = i\sigma_x$

$\sigma_z\sigma_x = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = i\begin{pmatrix}0&-i\\i&0\end{pmatrix} = i\sigma_y$

$-\sigma_x\sigma_z = -\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = -\begin{pmatrix}0&-1\\1&0\end{pmatrix} = i\begin{pmatrix}0&-i\\i&0\end{pmatrix} = i\sigma_y$

<u>Exercise #5</u> Here, we prove the identity:

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}),$$

where $\vec{A}$ and $\vec{B}$ are two vector operators whose components commute with those of $\vec{\sigma}$.

$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})$

$= [(\sigma_x,\sigma_y,\sigma_z)\cdot(A_x,A_y,A_z)][(\sigma_x,\sigma_y,\sigma_z)\cdot(B_x,B_y,B_z)]$

$= [\sigma_x A_x + \sigma_y A_y + \sigma_z A_z][\sigma_x B_x + \sigma_y B_y + \sigma_z B_z]$

$= \sigma_x A_x \sigma_x B_x + \sigma_x A_x \sigma_y B_y + \sigma_x A_x \sigma_z B_z$
$+ \sigma_y A_y \sigma_x B_x + \sigma_y A_y \sigma_y B_y + \sigma_y A_y \sigma_z B_z$
$+ \sigma_z A_z \sigma_x B_x + \sigma_z A_z \sigma_y B_y + \sigma_z A_z \sigma_z B_z$

(now use the fact the components of σ commute with those of $\vec{A}$ and $\vec{B}$)

$$= \sigma_x^2 A_x B_x + \sigma_x\sigma_y A_x B_y + \sigma_x\sigma_z A_x B_z$$
$$+ \sigma_y\sigma_x A_y B_x + \sigma_y^2 A_y B_y + \sigma_y\sigma_z A_y B_z$$
$$+ \sigma_z\sigma_x A_z B_x + \sigma_z\sigma_y A_z B_y + \sigma_z^2 A_z B_z$$

(now use equations (34))

$$= A_x B_x + i\sigma_z A_x B_y - i\sigma_y A_x B_z$$
$$- i\sigma_z A_y B_x + A_y B_y + i\sigma_x A_y B_z$$
$$+ i\sigma_y A_z B_x - i\sigma_x A_z B_y + A_z B_z$$
$$= A_x B_x + A_y B_y + A_z B_z + i\sigma_x (A_y B_z - A_z B_y) + i\sigma_y (A_z B_x - A_x B_z) + i\sigma_z (A_x B_y - A_y B_x)$$
$$= \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}) \quad \text{Q.E.D.}$$

Exercise #6

$$\alpha_x^2 = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_x^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Similarly, $\alpha_y^2 = \alpha_z^2 = 1$.

$$\beta^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

$$\alpha_x\alpha_y + \alpha_y\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = \begin{pmatrix} \sigma_x\sigma_y + \sigma_y\sigma_x & 0 \\ 0 & \sigma_x\sigma_y + \sigma_y\sigma_x \end{pmatrix} = 0$$

$$\alpha_x\alpha_z + \alpha_z\alpha_x = \begin{pmatrix} \sigma_x\sigma_z + \sigma_z\sigma_x & 0 \\ 0 & \sigma_x\sigma_z + \sigma_z\sigma_x \end{pmatrix} = 0$$

$$\alpha_y\alpha_z + \alpha_z\alpha_y = \begin{pmatrix} \sigma_y\sigma_z + \sigma_z\sigma_y & 0 \\ 0 & \sigma_y\sigma_z + \sigma_z\sigma_y \end{pmatrix} = 0$$

$$\alpha_x\beta + \beta\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_x + \sigma_x \\ \sigma_x - \sigma_x & 0 \end{pmatrix} = 0$$

$$\alpha_y\beta + \beta\alpha_y = \begin{pmatrix} 0 & -\sigma_y + \sigma_y \\ \sigma_y - \sigma_y & 0 \end{pmatrix} = 0$$

$$\alpha_z\beta + \beta\alpha_z = \begin{pmatrix} 0 & -\sigma_z + \sigma_z \\ \sigma_z - \sigma_z & 0 \end{pmatrix} = 0$$

We see that the Dirac matrices $\alpha_x, \alpha_y, \alpha_z, \beta$ all square to unity; different Dirac matrices anticommute. The Pauli matrices also square to unity; different Pauli matrices also anticommute. (eg $\sigma_x\sigma_y = -\sigma_y\sigma_x \Rightarrow \sigma_x\sigma_y + \sigma_y\sigma_x = 0$)

Exercise #7

$$\vec{\alpha}^\dagger = (\alpha_x, \alpha_y, \alpha_z)^\dagger$$
$$= (\alpha_x^\dagger, \alpha_y^\dagger, \alpha_z^\dagger)$$
$$= \left(\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}^\dagger, \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}^\dagger, \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}^\dagger\right)$$

$$= \left(\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}^{\dagger}, \begin{pmatrix} 0&0&0&-i\\ 0&0&i&0\\ 0&-i&0&0\\ i&0&0&0 \end{pmatrix}^{\dagger}, \begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix}^{\dagger} \right)$$

(remember, the dagger of a matrix is the complex conjugate of the transposed matrix... cf. footnote 22)

$$= \left(\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&-i\\ 0&0&i&0\\ 0&-i&0&0\\ i&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} 0&\sigma_x\\ \sigma_x&0 \end{pmatrix}, \begin{pmatrix} 0&\sigma_y\\ \sigma_y&0 \end{pmatrix}, \begin{pmatrix} 0&\sigma_z\\ \sigma_z&0 \end{pmatrix} \right)$$

$$= (\alpha_x, \alpha_y, \alpha_z)$$

$$= \vec{\alpha} \quad \text{Q.E.D.}$$

$$\beta^{\dagger} = \begin{pmatrix} 1&0\\ 0&-1 \end{pmatrix}^{\dagger} = \begin{pmatrix} 1&0\\ 0&-1 \end{pmatrix} = \beta \quad \text{Q.E.D.}$$

A necessary, but not sufficient, condition for an operator to be an "observable" in the quantum-mechanical sense of the term, is that the said operator be Hermitian. In turn, observables are the only Hermitian operators capable of having a physical interpretation. For more information on observables, see Albert Messiah, <u>Quantum Mechanics volume I</u>, pp. 257 – 260.

<u>Exercise $7\frac{1}{2}$</u> Equation (18a) was obtained by premultiplying both sides of the Klein-Gordon equation by the complex conjugate of the Klein-Gordon wavefunction; similarly, (57) was obtained by pre-multiplying

both sides of the Dirac equation by the Hermitian conjugate of the Dirac wave function. A similar argument applies to (18b) and (59). The manipulations for the Klein-Gordon and Dirac equations are therefore analogous, with complex conjugation being replaced by Hermitian conjugation.

Take Dirac equation:

(1) $$(i\partial_t + i\vec{\alpha}\cdot\nabla - \beta m)\Psi = 0$$

and pre-multiply by $\Psi^\dagger$:

(2) $$\Psi^\dagger(i\partial_t + i\vec{\alpha}\cdot\nabla - \beta m)\Psi = 0.$$

Now take the Hermitian conjugate of Dirac equation:

(3) $$-i\partial_t\Psi^\dagger - i\nabla\Psi^\dagger\cdot\vec{\alpha} - m\Psi^\dagger\beta = 0$$

(4) $$i\partial_t\Psi^\dagger + i\nabla\Psi^\dagger\cdot\vec{\alpha} + m\Psi^\dagger\beta = 0$$

and post-multiply by Ψ:

(5) $$i(\partial_t\Psi^\dagger)\Psi + i\nabla\Psi^\dagger\cdot\vec{\alpha}\Psi + m\Psi^\dagger\beta\Psi = 0$$

Now subtract (5) from (2):

(6) $$\Psi^\dagger i\partial_t\Psi + i\Psi^\dagger\vec{\alpha}\cdot\nabla\Psi - \Psi^\dagger\beta m\Psi - i(\partial_t\Psi^\dagger)\Psi - i\nabla\Psi^\dagger\cdot\vec{\alpha}\Psi - m\Psi^\dagger\beta\Psi = 0$$

which is not useful. Instead, add (5) to (2):

(7) $$\Psi^\dagger i\partial_t\Psi + i\Psi^\dagger\vec{\alpha}\cdot\nabla\Psi - \Psi^\dagger\beta m\Psi + i(\partial_t\Psi^\dagger)\Psi + i\nabla\Psi^\dagger\cdot\vec{\alpha}\Psi + m\Psi^\dagger\beta\Psi = 0$$

The two terms involving β cancel. Strike out "i" from all remaining terms, and re-arrange a little, to give:

(8) $$\Psi^\dagger\partial_t\Psi + (\partial_t\Psi^\dagger)\Psi + \Psi^\dagger\vec{\alpha}\cdot\nabla\Psi + \nabla\Psi^\dagger\cdot\vec{\alpha}\Psi = 0$$

(9) $$\partial_t(\Psi^\dagger\Psi) + \Psi^\dagger\vec{\alpha}\cdot\nabla\Psi + \nabla\Psi^\dagger\cdot\vec{\alpha}\Psi = 0$$

Now, from identity (61), we have:

(10) $$\nabla\cdot(\Psi^\dagger\vec{\alpha}\Psi) = \nabla\Psi^\dagger\cdot\vec{\alpha}\Psi + \Psi^\dagger\nabla\cdot\vec{\alpha}\Psi = \nabla\Psi^\dagger\cdot\vec{\alpha}\Psi + \Psi^\dagger\vec{\alpha}\cdot\nabla\Psi$$

Hence (9) becomes:

(11) $\partial_t(\Psi^\dagger \Psi) + \nabla \cdot (\Psi^\dagger \vec{\alpha} \Psi) = 0$

which may be compared to the continuity equation to conclude that:

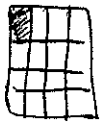
(12) $\rho = \Psi^\dagger \Psi$, (13) $\vec{j} = \Psi^\dagger \vec{\alpha} \Psi$.

Exercise #8 Claim: $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}$ $\quad \mu = 0,1,2,3$, $\nu = 0,1,2,3$

Since both the left and right sides are symmetric under the interchange $\mu \leftrightarrow \nu$, we need only check the cases where $\nu \geq \mu$.

$\mu = 0, \nu = 0$ $\quad \gamma_0\gamma_0 + \gamma_0\gamma_0 = 2\beta^2$

$= 2 = 2 g_{00}$ ✓



$\mu = 0, \nu = j = 1,2,3$ $\quad \gamma_0\gamma_j + \gamma_j\gamma_0$

$= \beta\beta\alpha_j + \beta\alpha_j\beta$ (by (54b))

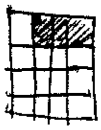
$= \alpha_j + \beta\alpha_j\beta$ (by (54d))

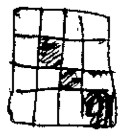
$= \alpha_j - \beta^2\alpha_j$ (by (54b))

$= \alpha_j - \alpha_j$

$= 0$

$= 2 g_{0j}$, $j = 1,2,3$ ✓





$\mu = \nu = j = 1,2,3$ $\quad \gamma_j\gamma_j + \gamma_j\gamma_j$

$= 2\gamma_j^2$

$= 2\beta\alpha_j\beta\alpha_j$

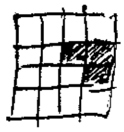
$= -2\beta\beta\alpha_j\alpha_j$ (by (54d))

$= -2\beta^2\alpha_j^2$

$= -2$ (by (54a) and (54b))

$= +2 g_{jj}$ ✓

~~[illegible]~~ $\nu > \mu > 0$ $\quad \mu = j \in \{1,2,3\}$, $\nu = j' \in \{1,2,3\}$, $j \neq j'$



$\gamma_j\gamma_{j'} + \gamma_{j'}\gamma_j = \beta\alpha_j\beta\alpha_{j'} + \beta\alpha_{j'}\beta\alpha_j$

$$= -\beta^2 \alpha_j \alpha_{j'} - \beta^2 \alpha_{j'} \alpha_j$$
$$= -\alpha_j \alpha_{j'} - \alpha_{j'} \alpha_j$$
$$= -(\alpha_j \alpha_{j'} + \alpha_{j'} \alpha_j)$$
$$= 0$$
$$= 2g_{jj'},\ j \neq j' > 0 \qquad \text{Q.E.D.}$$

Claim: $\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$. To see the validity of this claim, we separately check the cases $\mu = 0$ and $\mu > 0$.

$\underline{\underline{\mu = 0}}$

$$\gamma_0^\dagger \overset{?}{=} \gamma_0 \gamma_0 \gamma_0$$
$$\beta^\dagger \overset{?}{=} \beta\beta\beta$$
$$\beta^\dagger \overset{?}{=} \beta^2 \beta$$
$$\beta^\dagger \overset{?}{=} \beta \quad \checkmark$$

$\underline{\underline{\mu > 0}}$ Let $\mu = j \in \{1, 2, 3\}$

$$\gamma_j^\dagger \overset{?}{=} \gamma_0 \gamma_j \gamma_0, \quad j = 1, 2, 3$$
$$(\beta \alpha_j)^\dagger \overset{?}{=} \beta \beta \alpha_j \beta$$
$$(\beta \alpha_j)^\dagger \overset{?}{=} \beta^2 \alpha_j \beta$$
$$\alpha_j^\dagger \beta^\dagger \overset{?}{=} \alpha_j \beta$$

But $\beta^\dagger = \beta$. Post multiply both sides by β and use $\beta^2 = 1$ to get:

$$\alpha_j^\dagger \overset{?}{=} \alpha_j \quad \checkmark$$

$\underline{\underline{\text{Exercise}}}$ #9 Let $j = A$ or B. Then $\Psi_j(\vec{r}, t) = \tilde{\Psi}_j(\vec{r}) e^{-iE_0 t}$.

$$E\,\Psi_j(\vec{r}, t) = (i\partial_t - e\varphi)\,\tilde{\Psi}_j(\vec{r})\, e^{-iE_0 t}$$
$$= \tilde{\Psi}_j(\vec{r})\,(i\partial_t - e\varphi)\, e^{-iE_0 t}$$
$$= \tilde{\Psi}_j(\vec{r})\,(i(-iE_0) - e\varphi)\, e^{-iE_0 t}$$
$$= (E_0 - e\varphi)\,\tilde{\Psi}_j(\vec{r})\, e^{-iE_0 t}$$
$$= (E_0 - e\varphi)\,\Psi_j(\vec{r}, t).$$

Therefore the wavefunctions in (70) are indeed eigenfunctions of the energy operator.

The probability density of the wave-functions are:

$$P_j = |\Psi_j(\vec{r},t)|^2 \qquad j = A \text{ or } B$$
$$= |\Psi_j(\vec{r})e^{-iE_0 t}|^2$$
$$= |\Psi_j(\vec{r})|^2$$

which is independent of time.

Exercise #10 Consider equations (71), namely:

$$\begin{cases} \vec{\sigma}\cdot(i\nabla + e\vec{A})\Psi_B = (-E_0 + e\varphi + m)\Psi_A \\ \vec{\sigma}\cdot(i\nabla + e\vec{A})\Psi_A = (-E_0 + e\varphi - m)\Psi_B \end{cases} \quad (71)$$

Use the first of these equations to solve for Ψ_A:

$$\Rightarrow \Psi_A = \frac{1}{-E_0 + e\varphi + m}\,\vec{\sigma}\cdot(i\nabla + e\vec{A})\Psi_B.$$

Substitute into the second member of (71), to gives:

$$\vec{\sigma}\cdot(i\nabla + e\vec{A})\frac{1}{-E_0 + e\varphi + m}\vec{\sigma}\cdot(i\nabla + e\vec{A})\Psi_B$$
$$= (-E_0 + e\varphi - m)\Psi_B$$

which is the required result.

Exercise #11 We take as a starting point the vector identity: ~~[illegible]~~

$\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$, where ϕ is a differentiable scalar function of position and $\vec{A}$ is a differentiable vector ~~[illegible]~~ function of position. Since $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$, we may rewrite our identity as:

~~[illegible]~~

$$\nabla\times(\phi\vec{A}) = -\vec{A}\times\nabla\phi + \phi(\nabla\times\vec{A})$$
$$\nabla\times(\vec{A}\phi) = -\vec{A}\times\nabla\phi + (\nabla\times\vec{A})\phi$$
$$\nabla\times\vec{A}\,\phi = -\vec{A}\times\nabla\phi + (\nabla\times\vec{A})\phi$$

This is equivalent to the operator equation:

$$\nabla\times\vec{A} = -\vec{A}\times\nabla + (\nabla\times\vec{A})$$
$$\nabla\times\vec{A} + \vec{A}\times\nabla = (\nabla\times\vec{A}) \quad \text{Q.E.D.}$$

Exercise #12 When $\Psi_B \ll \Psi_A$, as it is in §5.5 (cf. eq. 89), then the normalisation condition 98:

$$\iiint (\Psi_A^\dagger \Psi_A + \Psi_B^\dagger \Psi_B)\,dx\,dy\,dz = 1$$

reduces to:

$$\iiint \Psi_A^\dagger \Psi_A \,dx\,dy\,dz \approx 1,$$

from which we see that Ψ_A is normalised (to within our approximation). Therefore we did not need to consider "Difficulty 2" in §5.5.

Exercise #13 Take the second of equations 90:

$$\vec{\sigma}\cdot\left(-i\hbar\nabla - \frac{e\vec{A}}{c}\right)\Psi_A = \frac{1}{c}(E_0 - e\varphi + mc^2)\Psi_B,$$

set $\vec{A}=0$:

$$-i\hbar\vec{\sigma}\cdot\nabla\Psi_A = \frac{1}{c}(E_0 - e\varphi + mc^2)\Psi_B$$

and then solve for Ψ_B:

$$\Rightarrow \Psi_B = \frac{-ic\hbar}{E_0 - e\varphi + mc^2}\,\vec{\sigma}\cdot\nabla\Psi_A$$

Exercise #14 From identity 31, we have:

$$(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = \vec{p}\cdot\vec{p} + i\vec{\sigma}\cdot(\vec{p}\times\vec{p})$$
$$= \vec{p}\cdot\vec{p} \quad \text{Q.E.D.}$$

Exercise #15 (112) $\Omega^{-1}\tilde{H}\Omega^{-1}\breve{\Psi} \approx \tilde{E}\Omega^{-2}\breve{\Psi}$

use (97b), (108), (109):

$$\left[1-\frac{\vec{p}^2}{8m^2c^2}\right]\left[\frac{-\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)+e\varphi\right]\left[1-\frac{\vec{p}^2}{8m^2c^2}\right]\breve{\Psi} = \tilde{E}\left[1-\frac{\vec{p}^2}{8m^2c^2}\right]^2\breve{\Psi}$$

Now expand all the brackets...

$$\left[1-\frac{\vec{p}^2}{8m^2c^2}\right]\left[\frac{-\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)+e\varphi+\frac{\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)\frac{\vec{p}^2}{8m^2c^2}-\frac{e\varphi\vec{p}^2}{8m^2c^2}\right]\breve{\Psi}$$

$$=\tilde{E}\left[1-\frac{\vec{p}^2}{4m^2c^2}+\frac{\vec{p}^4}{64m^4c^4}\right]\breve{\Psi}$$

$$\left[\begin{array}{l}\frac{-\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)+e\varphi+\frac{\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)\frac{\vec{p}^2}{8m^2c^2}-\frac{e\varphi\vec{p}^2}{8m^2c^2}\\ +\frac{\vec{p}^2\hbar^2}{16m^3c^2}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)-\frac{\vec{p}^2e\varphi}{8m^2c^2}-\frac{\vec{p}^2\hbar^2}{16m^3c^2}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)\frac{\vec{p}^2}{8m^2c^2}+\frac{e\varphi\vec{p}^4}{64m^4c^4}\end{array}\right]\breve{\Psi}$$

$$=\tilde{E}\left[1-\frac{\vec{p}^2}{4m^2c^2}+\frac{\vec{p}^4}{64m^4c^4}\right]\breve{\Psi}$$

Now strike out all terms which have c^{-3}, c^{-4}, etc... ie work to order c^{-2}.

$$\left[\begin{array}{l}\frac{-\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)\left(1-\frac{\tilde{E}-e\varphi}{2mc^2}\right)(\vec{\sigma}\cdot\nabla)+e\varphi+\frac{\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)(1)(\vec{\sigma}\cdot\nabla)\frac{\vec{p}^2}{8m^2c^2}-\frac{e\varphi\vec{p}^2}{8m^2c^2}\\ +\frac{\vec{p}^2\hbar^2}{16m^3c^2}(\vec{\sigma}\cdot\nabla)(1)(\vec{\sigma}\cdot\nabla)-\frac{\vec{p}^2e\varphi}{8m^2c^2}\end{array}\right]\breve{\Psi}$$

$$\approx\tilde{E}\left[1-\frac{\vec{p}^2}{4m^2c^2}\right]\breve{\Psi}$$

$$\left[\begin{array}{l}\frac{-\hbar^2}{2m}(\vec{\sigma}\cdot\nabla)^2+\frac{\hbar^2(\vec{\sigma}\cdot\nabla)}{4m^2c^2}(\tilde{E}-e\varphi)(\vec{\sigma}\cdot\nabla)+e\varphi+\frac{\hbar^2}{16m^3c^2}(\vec{\sigma}\cdot\nabla)^2\vec{p}^2-\frac{e}{8m^2c^2}\left(\varphi\vec{p}^2+\vec{p}^2\varphi\right)\\ +\frac{\vec{p}^2\hbar^2}{16m^3c^2}(\vec{\sigma}\cdot\nabla)^2\end{array}\right]\breve{\Psi}$$

$$=\tilde{E}\left[1-\frac{\vec{p}^2}{4m^2c^2}\right]\breve{\Psi}.$$

(Now, from 103, $(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^{\,2}$
$$\Rightarrow (\vec{\sigma}\cdot -i\hbar\nabla)^2 = \vec{p}^{\,2}$$
$$\Rightarrow -\hbar^2(\vec{\sigma}\cdot\nabla)^2 = \vec{p}^{\,2})$$

$$\left[\frac{\vec{p}^{\,2}}{2m} + \frac{\hbar^2(\vec{\sigma}\cdot\nabla)}{4m^2c^2}(\tilde{E}-e\varphi)(\vec{\sigma}\cdot\nabla) + e\varphi - \frac{\vec{p}^{\,4}}{16m^3c^2} - \frac{e}{8m^2c^2}[\varphi, \vec{p}^{\,2}]_+ - \frac{\vec{p}^{\,4}}{16m^3c^2}\right]\breve{\Psi}$$
$$= \tilde{E}\left[1 - \frac{\vec{p}^{\,2}}{4m^2c^2}\right]\breve{\Psi}$$

$$\left[\frac{p^2}{2m} + e\varphi - \frac{\vec{p}^{\,4}}{8m^3c^2} - \left[\frac{\vec{p}^{\,2}}{8m^2c^2}, e\varphi\right]_+ + \hbar^2\frac{(\vec{\sigma}\cdot\nabla)(\tilde{E}-e\varphi)(\vec{\sigma}\cdot\nabla)}{4m^2c^2}\right]\breve{\Psi}$$
$$= \tilde{E}\left[1 - \frac{\vec{p}^{\,2}}{4m^2c^2}\right]\breve{\Psi}$$

$$\left[\frac{\vec{p}^{\,2}}{2m} + e\varphi - \frac{\vec{p}^{\,4}}{8m^3c^2} - \left[\frac{\vec{p}^{\,2}}{8m^2c^2}, e\varphi\right]_+ - \frac{(\vec{\sigma}\cdot -i\hbar\nabla)(\tilde{E}-e\varphi)(\vec{\sigma}\cdot -i\hbar\nabla)}{4m^2c^2}\right]\breve{\Psi}$$
$$= \tilde{E}\left[1 - \frac{\vec{p}^{\,2}}{4m^2c^2}\right]\breve{\Psi}$$

$$\left[\frac{\vec{p}^{\,2}}{2m} + e\varphi - \frac{\vec{p}^{\,4}}{8m^3c^2} - \left[\frac{\vec{p}^{\,2}}{8m^2c^2}, e\varphi\right]_+ - \frac{1}{4m^2c^2}(\vec{\sigma}\cdot\vec{p})(\tilde{E}-e\varphi)(\vec{\sigma}\cdot\vec{p})\right]\breve{\Psi}$$
$$= \tilde{E}\left[1 - \frac{\vec{p}^{\,2}}{4m^2c^2}\right]\breve{\Psi}. \quad \text{Q.E.D.}$$

Exercise #16

~~$[A^2, B]_+ - 2ABA$~~
~~$= A^2B + BA^2 + 2ABA$~~

$$[A,[A,B]]$$
$$= A[A,B] - [A,B]A$$
$$= A(AB - BA) - (AB - BA)A$$
$$= AAB - ABA - ABA + BAA$$
$$= AAB + BAA - 2ABA$$
$$= [A^2, B]_+ - 2ABA \quad \text{Q.E.D.}$$

Exercise #17

$$[(\vec{\sigma}\cdot\vec{p}),[(\vec{\sigma}\cdot\vec{p}),\tilde{E}-e\varphi]]\Psi \quad \text{(use (121) for inner commutator)}$$
$$= [(\vec{\sigma}\cdot\vec{p}),-ie\hbar(\vec{\sigma}\cdot\vec{E})]\Psi$$
$$= -ie\hbar[(\vec{\sigma}\cdot\vec{p}),(\vec{\sigma}\cdot\vec{E})]\Psi$$
$$= -ie\hbar((\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{E})-(\vec{\sigma}\cdot\vec{E})(\vec{\sigma}\cdot\vec{p}))\Psi \quad \text{(now use (31))}$$
$$= -ie\hbar(\vec{p}\cdot\vec{E}+i\vec{\sigma}\cdot(\vec{p}\times\vec{E})-\vec{E}\cdot\vec{p}-i\vec{\sigma}\cdot(\vec{E}\times\vec{p}))\Psi$$
$$= (-ie\hbar)(-i\hbar)(\nabla\cdot\vec{E}+i\vec{\sigma}\cdot(\nabla\times\vec{E})-\vec{E}\cdot\nabla-i\vec{\sigma}\cdot(\vec{E}\times\nabla))\Psi$$
$$= -e\hbar^2(\nabla\cdot(\vec{E}\Psi)-\vec{E}\cdot\nabla\Psi+i\vec{\sigma}\cdot(\nabla\times\vec{E}\Psi-\vec{E}\times\nabla\Psi)) \quad \text{zero}$$
$$= -e\hbar^2(\cancel{\vec{E}\cdot\nabla\Psi}+(\nabla\cdot\vec{E})\Psi-\cancel{\vec{E}\cdot\nabla\Psi}+i\vec{\sigma}\cdot(\underbrace{-\vec{E}\times\nabla\Psi+(\nabla\times\vec{E})\Psi}_{\text{see solution to exercise \#11}}-\vec{E}\times\nabla\Psi))$$
$$= -e\hbar^2((\nabla\cdot\vec{E})\Psi-2i\vec{\sigma}\cdot(\vec{E}\times\nabla\Psi))$$
$$= \{-e\hbar^2(\nabla\cdot\vec{E})+2ie\hbar^2\vec{\sigma}\cdot(\vec{E}\times\nabla)\}\Psi$$
$$= \{-e\hbar^2(\nabla\cdot\vec{E})-2e\hbar\vec{\sigma}\cdot(\vec{E}\times -i\hbar\nabla)\}\Psi$$
$$= \{-e\hbar^2(\nabla\cdot\vec{E})-2e\hbar\vec{\sigma}\cdot(\vec{E}\times\vec{p})\}\Psi$$

Therefore: $[(\vec{\sigma}\cdot\vec{p}),[(\vec{\sigma}\cdot\vec{p}),\tilde{E}-e\varphi]] = -e\hbar^2(\nabla\cdot\vec{E})-2e\hbar\vec{\sigma}\cdot(\vec{E}\times\vec{p})$, which is the required result.

Exercise #18

$$\sqrt{(mc^2)^2+\vec{p}^2c^2}-mc^2$$
$$= mc^2\left(\sqrt{1+\frac{\vec{p}^2c^2}{(mc^2)^2}}-1\right)$$

Binomial approximation: $\left[(1+\theta)^a \simeq 1+a\theta+\frac{a(a-1)}{2}\theta^2 \text{ when } \theta \ll 1\right]$

$$\simeq mc^2\left(1+\frac{1}{2}\frac{\vec{p}^2c^2}{(mc^2)^2}+\frac{1}{2}\frac{1}{2}\times\left(\frac{1}{2}-1\right)\left(\frac{\vec{p}^2c^2}{(mc^2)^2}\right)^2-1\right)$$
$$= mc^2\left(\frac{\vec{p}^2c^2}{2m^2c^4}-\frac{\vec{p}^4c^4}{8m^4c^8}\right)+\dots$$
$$= \frac{\vec{p}^2}{2m}-\frac{\vec{p}^4}{8m^3c^2}+\dots \quad \text{Q.E.D.}$$

If we note the vector identity $\nabla \cdot \nabla \times \vec{C} = 0$, and compare it to the Maxwell equation $\nabla \cdot \vec{B} = \vec{0}$, we see that there exists a vector potential $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$. This proves equation (179a). If we now substitute this result into the Maxwell equation $\nabla \times \vec{E} = -c^{-1}\partial_t \vec{B}$, we obtain:

$$\nabla \times \vec{E} = -c^{-1}\partial_t \nabla \times \vec{A}$$
$$= -c^{-1}\nabla \times \partial_t \vec{A}$$

Hence $\nabla \times (\vec{E} + c^{-1}\partial_t \vec{A}) = 0$. Now, if we note the vector identity $\nabla \times \nabla \Phi = 0$, we see that $\vec{E} + c^{-1}\partial_t \vec{A} = -\nabla \Phi$. Hence $\vec{E} = -\nabla \Phi - c^{-1}\partial_t \vec{A}$, which proves (179b). By construction, then, the Maxwell equations $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -c^{-1}\partial_t \vec{B}$ are automatically satisfied when we work in terms of the potentials $\Phi, \vec{A}$ rather than the fields $\vec{E}, \vec{B}$.

Exercise #26 Suppose that we make a gauge transformation of the second kind on the potentials Φ and $\vec{A}$:

$$\Phi \to \Phi' = \Phi + c^{-1}\partial_t f, \quad \vec{A} \to \vec{A}' = \vec{A} - \nabla f.$$

Then the "new electric field" is:

$$\vec{E}' = -\nabla \Phi' - c^{-1}\partial_t \vec{A}' \quad \text{(by (179b))}$$
$$= -\nabla(\Phi + c^{-1}\partial_t f) - c^{-1}\partial_t(\vec{A} - \nabla f)$$
$$= -\nabla \Phi - c^{-1}\partial_t \vec{A} - \cancel{\nabla c^{-1}\partial_t f} + \cancel{c^{-1}\partial_t \nabla f}$$
$$= \vec{E} \quad \text{(by (179b))}.$$ Hence the electric field is unaltered.

The "new" magnetic field is:

$$\vec{B}' = \underbrace{\nabla \times \vec{A}'}_{\text{by (179a)}} = \nabla \times (\vec{A} - \nabla f) = \nabla \times \vec{A} - \underbrace{\nabla \times \nabla f}_{\text{zero}} = \vec{B}.$$

Hence the magnetic field is unaltered.

Since both the electric and magnetic field are unaltered under the gauge transformation, there is evidently some freedom - called "gauge freedom" in the choice of scalar and vector potentials. This freedom allows us to use a "gauge" which is most convenient for a given calculation. This choice of gauge has no physical significance in classical electrodynamics. (The situation changes in quantum mechanics - look up a discussion on the "Aharonov-Bohm" effect if you are interested in pursuing this point further.)

Exercise #27 Suppose that we are given potentials Φ and $\vec{A}$ that do not satisfy $\nabla\cdot\vec{A}=0$. We want to make a gauge transformation, using (180), so that the transformed potentials do satisfy $\nabla\cdot\vec{A}=0$. Thus:

$$\Phi \to \Phi' = \Phi + c^{-1}\partial_t f,\ \vec{A}\to\vec{A}'=\vec{A}-\nabla f,\ \nabla\cdot\vec{A}'=0.$$

The question therefore reduces to the problem of finding f so that the above equations are true. Now, since $\nabla\cdot\vec{A}\neq 0$, let $\nabla\cdot\vec{A}=g$, where g is known because $\vec{A}$ is known. Since we demand $\nabla\cdot\vec{A}'=0$, we have $\nabla\cdot(\vec{A}-\nabla f)=0$, ie: $\nabla\cdot\vec{A}-\nabla^2 f=0$. But $\nabla\cdot\vec{A}=g$, so that $\nabla^2 f$ is g. Any good text on the theory of partial differential equations will tell you how to solve the Poisson equation $\nabla^2 f=g$ for f, given g. But f is the required gauge function that leads to $\nabla\cdot\vec{A}'=0$. Hence one can always choose a gauge for the potentials such that the vector potential is divergence free, i.e. $\nabla\cdot\vec{A}'=0$.

Exercise #28 If we apply the Coulomb condition (185) to (190), we see that: $0=\nabla\cdot\vec{A}(\vec{x},t)=\nabla\cdot(\vec{A}_0\exp(i(\vec{k}\cdot\vec{x}-\omega t)))$. This is in the form: divergence of (scalar × vector). Note, in this context, the identity from vector analysis: $\nabla\cdot(f\vec{g})=\nabla f\cdot\vec{g}+f(\nabla\cdot\vec{g})$. If we identify the scalar f with $\exp(i(\vec{k}\cdot\vec{x}-\omega t))$, and the vector $\vec{g}$ with $\vec{A}_0$, we can proceed further:

$$0=[\nabla\exp(i(\vec{k}\cdot\vec{x}-\omega t))]\cdot\vec{A}_0+\exp(i(\vec{k}\cdot\vec{x}-\omega t))\underbrace{[\nabla\cdot\vec{A}_0]}_{\text{zero, because } \vec{A}_0 \text{ has no spatial dependence.}}$$

$$=i\vec{k}\exp(i(\vec{k}\cdot\vec{x}-\omega t))\cdot\vec{A}_0$$

$$=i\vec{k}\cdot\vec{A} \Rightarrow \vec{k}\cdot\vec{A}=0.\ \text{Q.E.D.}$$

Interpretation: the Coulomb condition implies that the polarisation $\vec{A}_0$ of the plane waves is perpendicular to the direction $\vec{k}$ of propagation.

Exercise #29 The requirement of "orthonormality" is by definition satisfied if both of the following hold: (a) the dot product of a given mode with the complex conjugate of a different mode, when integrated over the $L\times L\times L$ cube, gives zero. (b) The square modulus of a given mode, integrated over the cube, gives 1. Requirement (a) is the condition for orthogonality, while (b) is the requirement for normalisation.

Denote a given mode by: $\vec{A}_r(\vec{k},\vec{x})\equiv\frac{1}{\sqrt{V}}\vec{\varepsilon}_r(\vec{k})e^{i\vec{k}\cdot\vec{x}}$.

Then both (a) and (b) will be fulfilled if:

$$\int_0^L\int_0^L\int_0^L dx\,dy\,dz\,\vec{A}_r(\vec{k},\vec{x})\cdot\vec{A}^*_{r'}(\vec{k}',\vec{x})=\delta_{rr'}\delta_{\vec{k}\vec{k}'},\ \text{where}$$

δ denotes the Kronecker δ ($\delta_{ab}=0$ if $a\neq b$, $\delta_{ab}=1$ if $a=b$).

$$I = \int_0^L\int_0^L\int_0^L dx\,dy\,dz\, \vec{A}_r(\vec{k},\vec{x})\cdot\vec{A}_{r'}^{*}(\vec{k}',\vec{x})$$
$$= \int_0^L\int_0^L\int_0^L dx\,dy\,dz\, \frac{1}{\sqrt{V}}\vec{\varepsilon}_r(\vec{k})e^{i\vec{k}\cdot\vec{x}}\cdot\frac{1}{\sqrt{V}}\vec{\varepsilon}_{r'}^{*}(\vec{k}')e^{-i\vec{k}'\cdot\vec{x}}$$
$$= \frac{\vec{\varepsilon}_r(\vec{k})\cdot\vec{\varepsilon}_{r'}^{*}(\vec{k}')}{V}\int_0^L\int_0^L\int_0^L dx\,dy\,dz\,\exp\left(i(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x})\right)$$
assume $\vec{\varepsilon}_r(\vec{k})$ to be real, so we can drop the star.

case 1: $(\vec{k},r) = (\vec{k}',r')$

$$I = \frac{\vec{\varepsilon}_r(\vec{k})\cdot\vec{\varepsilon}_r(\vec{k})}{V}\int_0^L\int_0^L\int_0^L dx\,dy\,dz\,\exp(0)$$

Now use (195) $\Rightarrow$ $I = \frac{\delta_{rr}}{V}\times L^3 = 1$ which fulfils requirement (b).

case 2: $(\vec{k},r) \neq (\vec{k}',r')$

$$I = \frac{\vec{\varepsilon}_r(\vec{k})\cdot\vec{\varepsilon}_{r'}(\vec{k}')}{V}\int_0^L\int_0^L\int_0^L dx\,dy\,dz\,\exp\left(i(\vec{k}-\vec{k}')\cdot\vec{x}\right)$$

(now use (199), with $\vec{k} = \frac{2\pi}{L}(n_1,n_2,n_3)$, $\vec{k}' = \frac{2\pi}{L}(n_1',n_2',n_3')$) at least one member of each triplet must differ.

$$= \frac{\vec{\varepsilon}_r(\vec{k})\cdot\vec{\varepsilon}_{r'}(\vec{k}')}{V}\int_0^L\int_0^L\int_0^L dx\,dy\,dz\,\exp\left(\frac{i2\pi}{L}\left((n_1-n_1')x+(n_2-n_2')y+(n_3-n_3')z\right)\right)$$

$$= \frac{\vec{\varepsilon}_r(\vec{k})\cdot\vec{\varepsilon}_{r'}(\vec{k}')}{V}\underbrace{\int_0^L dx\,\exp\left(\frac{2\pi i}{L}(n_1-n_1')x\right)\int_0^L dy\,\exp\left(\frac{2\pi i}{L}(n_2-n_2')y\right)\times\int_0^L dz\,\exp\left(\frac{2\pi i}{L}(n_3-n_3')z\right)}_{\text{at least one of these integrals will be zero}}$$

$= 0$ which fulfils requirement (a). Hence the vector fields defined in (193) are orthonormal. Q.E.D.

Our next task is to show that the wave vectors in (199) ensure that the vector fields (193) satisfy periodic boundary conditions.

$$\vec{A}_r(\vec{k},\vec{x}) \equiv \frac{1}{\sqrt{V}}\vec{\varepsilon}_r(\vec{k})\exp(i\vec{k}\cdot\vec{x})$$

$n_1', n_2', n_3' = 0, \pm1, \pm2$ etc.

$$\Rightarrow \vec{A}_r(\vec{k}, x+n_1'L, y+n_2'L, z+n_3'L)$$
$$= \frac{1}{\sqrt{V}}\vec{\varepsilon}_r(\vec{k})\exp\left(i(k_x,k_y,k_z)\cdot(x+n_1'L, y+n_2'L, z+n_3'L)\right)$$

$$= \frac{1}{\sqrt{V}}\vec{\varepsilon}_r(\vec{k})\exp\left[i\{k_x(x+n_1'L)+k_y(y+n_2'L)+k_z(z+n_3'L)\}\right]$$

(now use (199))

$$= \frac{1}{\sqrt{V}}\vec{\varepsilon}_r(\vec{k})\exp\left[i\left\{\frac{2\pi n_1}{L}(x+n_1'L)+\frac{2\pi n_2}{L}(y+n_2'L)+\frac{2\pi n_3}{L}(z+n_3'L)\right\}\right]$$

$$= \frac{1}{\sqrt{V}}\vec{\varepsilon}_r(\vec{k})\exp\Big[i\underbrace{\left\{\frac{2\pi n_1 x}{L}+\frac{2\pi n_2 y}{L}+\frac{2\pi n_3 z}{L}\right\}}_{\vec{k}\cdot\vec{x}}\Big]$$

$$\times\exp\left[i\{2\pi n_1 n_1' + 2\pi n_2 n_2' + 2\pi n_3 n_3'\}\right]$$

(the top half of this equation is $\vec{A}_r(\vec{k},\vec{x})$)

$$= A_r(\vec{k},\vec{x})\times\exp\Big[i2\pi\underbrace{\{n_1n_1'+n_2n_2'+n_3n_3'\}}_{\text{this is an integer}}\Big]$$

$$= A_r(\vec{k},\vec{x})$$

Thus the vector fields in (193) satisfy the required periodic boundary condition. Q.E.D.

The vectors $\vec{\varepsilon}_1(\vec{k})$ and $\vec{\varepsilon}_2(\vec{k})$ can be interpreted as the two independent linear polarisation vectors of a plane wave with wavenumber $\vec{k}$. (cf. (95)).

Exercise #30 Substitute (196) into (198):

$$0 = (c^{-2}\partial_t^2 - \nabla^2)\sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\vec{\varepsilon}_r(\vec{k})\left(a_r(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} + a_r^*(\vec{k},t)e^{-i\vec{k}\cdot\vec{x}}\right)$$

$$= \sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\vec{\varepsilon}_r(\vec{k})\left[(c^{-2}\partial_t^2-\nabla^2)a_r(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} + (c^{-2}\partial_t^2-\nabla^2)a_r^*(\vec{k},t)e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$= \sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\vec{\varepsilon}_r(\vec{k})\left[\{(c^{-2}\partial_t^2 - i^2\|\vec{k}\|^2)a_r(\vec{k},t)\}e^{i\vec{k}\cdot\vec{x}} + \{(c^{-2}\partial_t^2 - i^2\|\vec{k}\|^2)a_r^*(\vec{k},t)\}e^{-i\vec{k}\cdot\vec{x}}\right]$$

The terms in braces are the Fourier coefficients of the Fourier series expansion of zero; therefore these braced terms are zero:

$$(c^{-2}\partial_t^2 + \|\vec{k}\|^2)a_r(\vec{k},t) = 0$$

[$\omega_k = c\|\vec{k}\|$ (see line after (196)) $\Rightarrow \|\vec{k}\|^2c^2 = \omega_k^2$]

$$\partial_t^2 a_r(\vec{k},t) = -\|\vec{k}\|^2c^2\, a_r(\vec{k},t)$$

$$\partial_t^2 a_r(\vec{k},t) = -\omega_k^2 a_r(\vec{k},t)$$

which is the required result. Q.E.D.

Exercise #31

$$[\hat{a},\hat{a}^+] \overset{(202)}{=} \left[\frac{m\omega\hat{q}+i\hat{p}}{\sqrt{2\hbar m\omega}}, \frac{m\omega\hat{q}-i\hat{p}}{\sqrt{2\hbar m\omega}}\right]$$

$$= \frac{1}{2\hbar m\omega}[m\omega\hat{q}+i\hat{p}, m\omega\hat{q}-i\hat{p}]$$

$$= \frac{1}{2\hbar m\omega}\left\{(m\omega\hat{q}+i\hat{p})(m\omega\hat{q}-i\hat{p}) - (m\omega\hat{q}-i\hat{p})(m\omega\hat{q}+i\hat{p})\right\}$$

$$= \frac{1}{2\hbar m\omega}\left\{\cancel{m^2\omega^2\hat{q}^2} - im\omega\hat{q}\hat{p} + im\omega\hat{p}\hat{q} + \cancel{\hat{p}^2} - \cancel{m^2\omega^2\hat{q}^2} - im\omega\hat{q}\hat{p} + im\omega\hat{p}\hat{q} - \cancel{\hat{p}^2}\right\}$$

$$= \frac{im\omega}{2\hbar m\omega}\left\{2\hat{p}\hat{q} - 2\hat{q}\hat{p}\right\}$$

$$= \frac{i}{\hbar}[\hat{p},\hat{q}] = \frac{-i}{\hbar}[\hat{q},\hat{p}] \overset{(201)}{=} \frac{-i}{\hbar}\times i\hbar = 1 \quad \text{Q.E.D.}$$

Exercise #32

$$\hat{H} \overset{(200)}{=} \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2$$

In order to proceed further, we need to obtain expressions for $\hat{p}$ and $\hat{q}$ in terms of $\hat{a}$ and $\hat{a}^+$. To do this, note that adding each of equations (202) leads to:

(α) $$(\hat{a}+\hat{a}^+)\sqrt{2\hbar m\omega} = 2m\omega\hat{q} \Rightarrow \hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)$$

Similarly, subtracting each of equations (202) gives:

(β) $$(\hat{a}-\hat{a}^+)\sqrt{2\hbar m\omega} = 2i\hat{p} \Rightarrow \hat{p} = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^+)$$

Now substitute (α) and (β) into (200), to give:

$$\hat{H} = \frac{-1}{2m}\left(\frac{\hbar m\omega}{2}\right)(\hat{a}-\hat{a}^+)^2 + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)(\hat{a}+\hat{a}^+)^2$$

$$= -\frac{\hbar\omega}{4}(\hat{a}-\hat{a}^+)^2 + \frac{\hbar\omega}{4}(\hat{a}+\hat{a}^+)^2$$

$$= \frac{\hbar\omega}{4}\left((\hat{a}+\hat{a}^+)^2 - (\hat{a}-\hat{a}^+)^2\right)$$

$$= \frac{\hbar\omega}{4}\left(\cancel{\hat{a}^2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \cancel{\hat{a}^{+2}} - \cancel{\hat{a}^2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} - \cancel{\hat{a}^{+2}}\right)$$

(γ) $$= \frac{\hbar\omega}{2}(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$$

Now, by (203), we have $\hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + 1$. Hence (γ) becomes:

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^+\hat{a} + 1 + \hat{a}^+\hat{a}) = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) \quad \text{Q.E.D.}$$

Exercise #33 We begin with the following commutator formula:

(α) $[\hat{A}\hat{B},\hat{C}]=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}$,

which can be proven by expanding out the right handside:

(β) $\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}=\hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B})+(\hat{A}\hat{C}-\hat{C}\hat{A})\hat{B}$

$$=\hat{A}\hat{B}\hat{C}-\cancel{\hat{A}\hat{C}\hat{B}}+\cancel{\hat{A}\hat{C}\hat{B}}-\hat{C}\hat{A}\hat{B}$$

$$=[\hat{A}\hat{B},\hat{C}]$$

We now use formula (α) to evaluate two different commutators:

(γ) $[\hat{N},\hat{a}]\overset{205}{=}[\hat{a}^+\hat{a},\hat{a}]\overset{\alpha}{=}\hat{a}^+\underbrace{[\hat{a},\hat{a}]}_{0}+\underbrace{[\hat{a}^+,\hat{a}]}_{-1,\text{ by } 203}\hat{a}=-\hat{a}$

(δ) $[\hat{N},\hat{a}^+]\overset{205}{=}[\hat{a}^+\hat{a},\hat{a}^+]\overset{\alpha}{=}\hat{a}^+\underbrace{[\hat{a},\hat{a}^+]}_{1,\text{ by } 203}+\underbrace{[\hat{a}^+,\hat{a}^+]}_{0}\hat{a}=\hat{a}^+$

Hence $\hat{N}\hat{a}|\beta\rangle\overset{\gamma}{=}(\hat{a}\hat{N}-\hat{a})|\beta\rangle$

$$=(\hat{a}\beta-\hat{a})|\beta\rangle$$

$$=(\beta-1)\hat{a}|\beta\rangle \quad \text{Q.E.D}$$

and: $\hat{N}\hat{a}^+|\beta\rangle\overset{\delta}{=}(\hat{a}^+\hat{N}+\hat{a}^+)\hat{a}|\beta\rangle=(\hat{a}^+\beta+\hat{a}^+)\hat{a}|\beta\rangle=(\beta+1)\hat{a}^+|\beta\rangle$ Q.E.D.

Exercise #34 We need to show that the states $|n\pm1\rangle$ are normalised, i.e. that $\langle n\pm1|n\pm1\rangle=1$.

(a) From (218a), $|n-1\rangle=\frac{1}{\sqrt{n}}\hat{a}|n\rangle$. The Hermitian conjugate of this equation is $\langle n-1|=\langle n|\hat{a}^+\frac{1}{\sqrt{n}}$. Hence:

$$\langle n-1|n-1\rangle=\langle n|\hat{a}^+\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\hat{a}|n\rangle\overset{205}{=}\frac{1}{n}\langle n|\hat{N}|n\rangle\overset{216}{=}\frac{1}{n}\langle n|n|n\rangle=\langle n|n\rangle\overset{217}{=}1 \quad \text{Q.E.D}$$

Similarly,

$$\langle n+1|n+1\rangle=\langle n|\frac{1}{\sqrt{n+1}}\hat{a}\frac{1}{\sqrt{n+1}}\hat{a}^+|n\rangle=\frac{1}{n+1}\langle n|\hat{a}\hat{a}^+|n\rangle$$

$$\overset{203}{=}\frac{1}{n+1}\langle n|1+\hat{a}^+\hat{a}|n\rangle$$

$$\overset{205}{=}\frac{1}{n+1}\langle n|1+\hat{N}|n\rangle$$

$$\overset{216}{=}\frac{1}{n+1}\langle n|1+n|n\rangle$$

$$=\frac{n+1}{n+1}\langle n|n\rangle=1 \quad \text{Q.E.D.}$$

Exercise #35

$$\frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle=\frac{(\hat{a}^+)^{n-1}\hat{a}^+|0\rangle}{\sqrt{n!}}$$

$$\overset{218b}{=}\frac{(\hat{a}^+)^{n-1}\sqrt{0+1}|1\rangle}{\sqrt{n!}}$$

$$=\frac{\sqrt{1}(\hat{a}^+)^{n-2}\hat{a}^+|1\rangle}{\sqrt{n!}}\overset{218b}{=}\frac{\sqrt{1}(\hat{a}^+)^{n-2}\sqrt{2}|2\rangle}{\sqrt{n!}}=\frac{\sqrt{1}\sqrt{2}(\hat{a}^+)^{n-3}\sqrt{3}|3\rangle}{\sqrt{n!}}$$

$$= \frac{\sqrt{1}\sqrt{2}\sqrt{3}\,(a^\dagger)^{n-4}\sqrt{4}\,|4\rangle}{\sqrt{n!}} = \cdots = \frac{\sqrt{n}\sqrt{n-1}\cdots\sqrt{1}}{\sqrt{n!}}|n\rangle = |n\rangle$$

Q.E.D.

Exercise #36

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle_S \overset{(223)}{=} i\hbar\frac{d}{dt}\hat{u}(t,t_0)|\Psi(t=t_0)\rangle_S$$

$$= i\hbar\left\{\frac{d}{dt}\hat{u}(t,t_0)\right\}|\Psi(t=t_0)\rangle_S$$

$$\overset{(229)}{=} i\hbar\left\{\frac{d}{dt}e^{-i(t-t_0)\hat{H}/\hbar}\right\}|\Psi(t=t_0)\rangle_S$$

$$= i\hbar\left(\frac{-i\hat{H}}{\hbar}\right)\underbrace{e^{-i(t-t_0)\hat{H}/\hbar}|\Psi(t=t_0)\rangle_S}_{|\Psi(t)\rangle_S}$$

$$= \hat{H}|\Psi(t)\rangle_S \quad \text{Q.E.D.}$$

The time evolution operator "winds the clock forward"; when applied to the schrodinger wave function at $t=t_0$, the operator $\hat{u}(t,t_0)$ yields the wave function at time t. Cf. (223).

$$\hat{u} = e^{-i(t-t_0)\hat{H}/\hbar}, \quad \hat{u}^\dagger = e^{+i(t-t_0)\hat{H}/\hbar}.$$

Now, we invoke the Campbell-Baker-Hausdorff theorem, which we shall not prove:

If $\hat{A}$ and $\hat{B}$ are operators such that $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$, then $\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-\frac{1}{2}[\hat{A},\hat{B}])$.

Let $\hat{A} = -i(t-t_0)\hat{H}/\hbar$, $\hat{B} = +i(t-t_0)\hat{H}/\hbar$, then $[\hat{A},\hat{B}]=0$ and so, for the case where $\hat{A}$ and $\hat{B}$ commute, $\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})$. Hence:

$$\hat{u}\hat{u}^\dagger = e^{-i(t-t_0)\hat{H}/\hbar}\,e^{+i(t-t_0)\hat{H}/\hbar}$$

$$= \exp\left(-i(t-t_0)\hat{H}/\hbar + i(t-t_0)\hat{H}/\hbar\right) = e^0 = I$$

Similarly, $\hat{u}^\dagger\hat{u} = I$. Q.E.D.

Exercise #37 Suppose that $[\hat{A}^H, \hat{B}^H] = \hat{C}^H$ for three operators in the Heisenberg picture. Now use (226b) to transform this commutation relation into:

$$[\hat{U}^+\hat{A}^S\hat{U}, \hat{U}^+\hat{B}^S\hat{U}] = \hat{U}^+\hat{C}^S\hat{U}.$$ Now expand the left side.

$$\hat{U}^+\hat{A}^S\underbrace{\hat{U}\hat{U}^+}_{I\text{, by previous exercise}}\hat{B}^S\hat{U} - \hat{U}^+\hat{B}^S\underbrace{\hat{U}\hat{U}^+}_{I}\hat{A}^S\hat{U} = \hat{U}^+\hat{C}^S\hat{U}$$

Pre-multiply both sides by $\hat{U}$ and post-multiply by $\hat{U}^+$:

$$\underbrace{\hat{U}\hat{U}^+}_{I}\hat{A}^S\hat{B}^S\underbrace{\hat{U}\hat{U}^+}_{I} - \underbrace{\hat{U}\hat{U}^+}_{I}\hat{B}^S\hat{A}^S\underbrace{\hat{U}\hat{U}^+}_{I} = \underbrace{\hat{U}\hat{U}^+}_{I}\hat{C}^S\underbrace{\hat{U}\hat{U}^+}_{I}$$

Hence $\hat{A}^S\hat{B}^S - \hat{B}^S\hat{A}^S = \hat{C}^S$, i.e. $[\hat{A}^S, \hat{B}^S] = \hat{C}^S$. Thus the transformation from Heisenberg to Schrödinger picture leaves the commutation relation $[\hat{A}, \hat{B}] = \hat{C}$ invariant. A similar calculation shows that transforming the other way, i.e. from Schrödinger to Heisenberg pictures, also leaves the said commutation relation invariant. Q.E.D.

$$\langle \Psi_A^{(H)} | \hat{O}^H | \Psi_B^{(H)} \rangle$$
$$\overset{(226)}{=} \underbrace{\{\langle \Psi_A^{(S)} | \hat{U}\}}_{\text{Hermitian conjugate of (226a)}} \hat{U}^+\hat{O}^S\hat{U}\{\hat{U}^+ | \Psi_B^{(S)} \rangle$$
$$= \langle \Psi_A^{(S)} | \underbrace{\hat{U}\hat{U}^+}_{I}\hat{O}^S\underbrace{\hat{U}\hat{U}^+}_{I} | \Psi_B^{(S)} \rangle = \langle \Psi_A^{(S)} | \hat{O}^S | \Psi_B^{(S)} \rangle$$

Thus the matrix element is invariant under the transformation between pictures.

Exercise #38 From the Heisenberg equation of motion (239), we have:

$$i\hbar \frac{d}{dt}\hat{a}^H(t) = [\hat{a}^H(t), \hat{H}_{\text{oscillator}}]$$ drop this subscript for rest of calculation!

$$= [\hat{a}(t), \hat{H}]$$ drop superscript

$$= [\hat{a}(t), \hbar\omega(\hat{a}^+(t)\hat{a}(t) + \tfrac{1}{2})]$$ by (204)

drop the "t"

$$= \hbar\omega[\hat{a}, \hat{a}^+\hat{a}]$$

$$= \hbar\omega\{\hat{a}^+\underbrace{\cancel{[\hat{a}, \hat{a}]}}_{\text{zero}} + \underbrace{[\hat{a}, \hat{a}^+]}_{1\text{, by (203)}}\hat{a}\} = \hbar\omega\hat{a}$$

Hence $\frac{d}{dt}\hat{a}^H(t) = \frac{\hbar\omega}{i\hbar}\hat{a}^H(t) = -i\omega\hat{a}^H(t)$

$$\Rightarrow \frac{d^2}{dt^2}\hat{a}^H(t) = (-i\omega)^2\hat{a}^H(t) = -\omega^2\hat{a}^H(t) \quad \text{Q.E.D.} \quad (\alpha)$$

The fact that $\hat{a}^H(t) = e^{-i\omega t}$ is a solution to (α) can be verified by direct substitution.

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Exercise #39 In the Coulomb gauge given by (185) $\nabla\cdot\vec{A} = 0$, both the electric field and the magnetic field can be derived from the vector potential using (189a) and (189b).

Electric field

$$(189b)\ \vec{E} = -c^{-1}\partial_t\vec{A}$$

$$\stackrel{(237)}{=} -c^{-1}\partial_t\sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\,\vec{\varepsilon}_r(\vec{k})\left[\hat{a}_r(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \hat{a}_r^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right]$$

$$= -c^{-1}\sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\,\vec{\varepsilon}_r(\vec{k})\left[\hat{a}_r(\vec{k})(-i\omega_k)e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \hat{a}_r^\dagger(\vec{k})(i\omega_k)e^{-i(\vec{k}\cdot\vec{x}-\omega_k}\right.$$

$$= i\sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar\omega_k}{2V}}\,\vec{\varepsilon}_r(\vec{k})\left[\hat{a}_r(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} - \hat{a}_r^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right]$$

Magnetic field

$$(189a)\ \vec{B} = \nabla\times\vec{A}$$

$$\stackrel{(237)}{=} \nabla\times\sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\,\vec{\varepsilon}_r(\vec{k})\left(\hat{a}_r(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \hat{a}_r^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right)$$

$$(\alpha)\quad = \sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\,\nabla\times\left[\vec{\varepsilon}_r(\vec{k})\left(\hat{a}_r(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \hat{a}_r^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right)\right]$$

Now, $\nabla\times(A\vec{B}) = \nabla A\times\vec{B} + A\nabla\times\vec{B}$

$\Rightarrow \nabla\times(\vec{B}A) = -\vec{B}\times\nabla A + (\nabla\times\vec{B})A$. For the above example, the vector $\vec{B} \equiv \vec{\varepsilon}_r(\vec{k})$ has no position dependence, so $(\nabla\times\vec{B}) = \vec{0}$ and our identity becomes $\nabla\times(\vec{B}A) = -\vec{B}\times\nabla A$. Thus equation (α) becomes:

$$\vec{B} = \sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\left[-\vec{\varepsilon}_r(\vec{k})\times\nabla\left(\hat{a}_r(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)}\right) - \vec{\varepsilon}_r(\vec{k})\times\nabla\left(\hat{a}_r^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k}\right.\right.$$

$$= \sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\left[-\vec{\varepsilon}_r(\vec{k})\times\left(\hat{a}_r(\vec{k})(i\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)}\right) - \vec{\varepsilon}_r(\vec{k})\times\left(\hat{a}_r^\dagger(\vec{k})(-i\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right.\right.$$

(Now factor out i, and use fact that $-\vec{\varepsilon}_r(\vec{k})\times\vec{k} = \vec{k}\times\vec{\varepsilon}_r(\vec{k})$)

$$= i\sum_{\vec{k}}\sum_r\sqrt{\frac{\hbar c^2}{2V\omega_k}}\left(\vec{k}\times\vec{\varepsilon}_r(\vec{k})\right)\left[\hat{a}_r(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} - \hat{a}_r^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right]$$

which is the required result.

Exercise #40

$$\langle n-1|\hat{\cos}\phi|n\rangle \overset{257a}{=} \langle n-1|\tfrac{1}{2}(e^{i\hat{\phi}}+e^{-i\hat{\phi}})|n\rangle$$

$$= \tfrac{1}{2}\langle n-1|\underbrace{e^{i\hat{\phi}}|n\rangle}_{=(1-\delta_{n0})|n-1\rangle,\text{ by (253a)}} + \tfrac{1}{2}\langle n-1|\underbrace{e^{-i\hat{\phi}}|n\rangle}_{=|n+1\rangle,\text{ by (253b)}}$$

$$= \tfrac{1}{2}(1-\delta_{n0})\underbrace{\langle n-1|n-1\rangle}_{\to 1} + \tfrac{1}{2}\underbrace{\langle n-1|n+1\rangle}_{\to 0}$$

$= \tfrac{1}{2}(1-\delta_{n0})$ ... But the fact that $\langle n-1|$ is a bra means $n \geq 1 \Rightarrow \delta_{n0}=0$

$= \tfrac{1}{2}$ Q.E.D.

$$\langle n-1|\hat{\sin}\phi|n\rangle \overset{257b}{=} \langle n-1|\tfrac{1}{2i}(e^{i\hat{\phi}}-e^{-i\hat{\phi}})|n\rangle$$

$$= \tfrac{1}{2i}\langle n-1|\underbrace{e^{i\hat{\phi}}|n\rangle} - \tfrac{1}{2i}\langle n-1|\underbrace{e^{-i\hat{\phi}}|n\rangle}$$

same tricks as:

$$= \tfrac{1}{2i}\langle n-1|(1-\delta_{n0})|n-1\rangle - \tfrac{1}{2i}\underbrace{\langle n-1|n+1\rangle}_{\to 0}$$

0 since $n \geq 1$

$$= \frac{(1-\cancel{\delta_{n0}})}{2i}\langle n-1|n-1\rangle$$

$= \frac{1}{2i}$ Q.E.D.

Calculations of $\langle n|\hat{\cos}\phi|n-1\rangle$ and $\langle n|\hat{\sin}\phi|n-1\rangle$ are analogous.

Exercise #41

$$[\hat{\cos}\phi, \hat{\sin}\phi] = \hat{\cos}\phi\,\hat{\sin}\phi - \hat{\sin}\phi\,\hat{\cos}\phi$$

$$= \tfrac{1}{4i}\left[(e^{i\hat{\phi}}+e^{-i\hat{\phi}})(e^{i\hat{\phi}}-e^{-i\hat{\phi}}) - (e^{i\hat{\phi}}-e^{-i\hat{\phi}})(e^{i\hat{\phi}}+e^{-i\hat{\phi}})\right]$$

$$= \tfrac{1}{4i}\Big[\cancel{(e^{i\hat{\phi}})^2} - e^{i\hat{\phi}}e^{-i\hat{\phi}} + e^{-i\hat{\phi}}e^{i\hat{\phi}} + \cancel{(e^{-i\hat{\phi}})^2} - \cancel{(e^{i\hat{\phi}})^2}$$

$$- e^{i\hat{\phi}}e^{-i\hat{\phi}} + e^{-i\hat{\phi}}e^{i\hat{\phi}} + \cancel{(e^{-i\hat{\phi}})^2}\Big]$$

$$= \tfrac{2}{4i}[e^{-i\hat{\phi}}e^{i\hat{\phi}} - e^{i\hat{\phi}}e^{-i\hat{\phi}}]$$ ... now use equations (252)

$$= \tfrac{1}{2i}\left[\hat{a}^{+}\frac{1}{\sqrt{\hat{n}+1}}\frac{1}{\sqrt{\hat{n}+1}}\hat{a} - \frac{1}{\sqrt{\hat{n}+1}}\hat{a}\hat{a}^{+}\frac{1}{\sqrt{\hat{n}+1}}\right]$$

$\hat{a}\hat{a}^{+} = \hat{a}^{+}\hat{a}+1$, by (203) $= \hat{n}+1$

$$= \frac{1}{2i}\left[\hat{a}^{+}\frac{1}{\hat{n}+1}\hat{a} - \frac{1}{\sqrt{\hat{n}+1}}\,\hat{n}+1\,\frac{1}{\sqrt{\hat{n}+1}}\right]$$

$$= \frac{1}{2i}\left[\hat{a}^{+}\frac{1}{\hat{n}+1}\hat{a} - 1\right] \quad \text{Q.E.D.}$$

The matrix elements of the commutator are $\langle m|[\hat{\cos}\phi, \hat{\sin}\phi]|n\rangle$, with $m, n = 0, +1, +2, +3$ etc. We separately consider the cases $n=0$ and $n \neq 0$.

$\underline{\underline{n=0}}$ $\langle m|[\hat{\cos}\phi, \hat{\sin}\phi]|0\rangle = \frac{1}{2i}\langle m|\hat{a}^{+}\frac{1}{\hat{n}+1}\hat{a} - 1|0\rangle$

$$= \frac{1}{2i}\Big(\langle m|\hat{a}^{+}\frac{1}{\hat{n}+1}\underbrace{\hat{a}|0\rangle}_{\to 0,\text{ because } \hat{a}|0\rangle = 0} - \underbrace{\langle m|0\rangle}_{\delta_{m0}}\Big)$$

$$= -\frac{\delta_{m0}}{2i}$$

$\Rightarrow$ For the case $n=0$, the only nonzero matrix element is

$$\langle 0|[\hat{\cos}\phi, \hat{\sin}\phi]|0\rangle = -\frac{1}{2i}$$

$\underline{\underline{n \neq 0}}$ $\langle m|[\hat{\cos}\phi, \hat{\sin}\phi]|n\rangle = \frac{1}{2i}\langle m|\hat{a}^{+}\frac{1}{\hat{n}+1}\hat{a} - 1|n\rangle, \; n \neq 0$

$$= \frac{1}{2i}\Big(\langle m|\hat{a}^{+}\frac{1}{\hat{n}+1}\underbrace{\hat{a}|n\rangle}_{\to \sqrt{n}|n-1\rangle,\text{ by (218 a)}} - \underbrace{\langle m|n\rangle}_{\to \delta_{mn}}\Big), \; n \neq 0$$

$$= \frac{1}{2i}\Big(\langle m|\hat{a}^{+}\frac{1}{\hat{n}+1}\sqrt{n}|n-1\rangle - \delta_{mn}\Big), \; n \neq 0$$

$$= \frac{1}{2i}\Big(\langle m|\hat{a}^{+}\frac{1}{(n-1)+1}\sqrt{n}|n-1\rangle - \delta_{mn}\Big), \; n \neq 0$$

$$= \frac{1}{2i}\Big(\frac{1}{\sqrt{n}}\langle m|\underbrace{\hat{a}^{+}|n-1\rangle}_{\to \text{now use (218b)}} - \delta_{mn}\Big), \; n \neq 0$$

$$= \frac{1}{2i}\Big(\frac{1}{\sqrt{n}}\langle m|\sqrt{n}|n\rangle - \delta_{mn}\Big) \; , n \neq 0$$

$$= \frac{1}{2i}\Big(\frac{\sqrt{n}}{\sqrt{n}}\delta_{mn} - \delta_{mn}\Big) = 0, \; n \neq 0$$

$\Rightarrow$ The only nonzero matrix element of the commutator $[\hat{\cos}\phi, \hat{\sin}\phi]$ is $\langle 0|[\hat{\cos}\phi, \hat{\sin}\phi]|0\rangle = -1/2i$.

Exercise #42 (a) we need to show that (268) is indeed a solution to the defining equation (267a) of the coherent states as the eigenfunctions of the destruction operator:

(267a) $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

Now use (268) $\Rightarrow$
$$\hat{a}|\alpha\rangle = \hat{a}\, e^{-\frac{1}{2}\|\alpha\|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$
$$= e^{-\frac{1}{2}\|\alpha\|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \hat{a}|n\rangle \quad \text{... now use (218a)}$$
$$= e^{-\frac{1}{2}\|\alpha\|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\,|n-1\rangle$$
$$= e^{-\frac{1}{2}\|\alpha\|^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle \quad \text{... now let } m=n-1$$
$$= e^{-\frac{1}{2}\|\alpha\|^2} \sum_{m=0}^{\infty} \frac{\alpha^{m+1}}{\sqrt{m!}} |m\rangle$$
$$= \alpha\, e^{-\frac{1}{2}\|\alpha\|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle = \alpha|\alpha\rangle \quad \text{Q.E.D.}$$

(b) Since the Fock states are "complete", we can expand $|\alpha\rangle$ in terms of them:
$$|\alpha\rangle = \sum_{m=0}^{\infty} \alpha_m |m\rangle,$$
where $|m\rangle$ are the Fock states. The problem therefore reduces to one of finding the coefficients α_m in the expansion. Substitute our expansion into the defining relation (267 a) for the coherent states:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$
$$\hat{a}\sum_{m=0}^{\infty}\alpha_m|m\rangle = \alpha\sum_{m=0}^{\infty}\alpha_m|m\rangle$$
$$\sum_{m=0}^{\infty}\alpha_m \underbrace{\hat{a}|m\rangle}_{\sqrt{m}\,|m-1\rangle} = \alpha\sum_{m=0}^{\infty}\alpha_m|m\rangle$$
$$\sum_{m=0}^{\infty}\alpha_m \sqrt{m}\,|m-1\rangle = \alpha\sum_{m=0}^{\infty}\alpha_m|m\rangle$$
$$\sum_{m=1}^{\infty}\alpha_m \sqrt{m}\,|m-1\rangle = \alpha\sum_{m=0}^{\infty}\alpha_m|m\rangle$$

On the left side, let $m-1=n$.
" " right ", let $m=n$.
$$\sum_{n=0}^{\infty}\alpha_{n+1}\sqrt{n+1}\,|n\rangle = \alpha\sum_{n=0}^{\infty}\alpha_n|n\rangle.$$
$$\Rightarrow 0 = \sum_{n=0}^{\infty}\left(\alpha_{n+1}\sqrt{n+1} - \alpha\alpha_n\right)|n\rangle$$
Now premultiply both sides by the bra $\langle m|$.
$$\langle m|0 = \langle m|\sum_{n=0}^{\infty}\left(\alpha_{n+1}\sqrt{n+1} - \alpha\alpha_n\right)|n\rangle$$
$$0 = \sum_{n=0}^{\infty}\left(\alpha_{n+1}\sqrt{n+1} - \alpha\alpha_n\right)\underbrace{\langle m|n\rangle}_{\delta_{mn}}$$
$$= \alpha_{m+1}\sqrt{m+1} - \alpha\alpha_m$$

This equation, $\alpha_{m+1}\sqrt{m+1} - \alpha\alpha_m = 0$, is a "recursion relation" which allows us to determine α_{m+1} from α_m. Let α_0 be given. Then our recursion relation, which may be written as:

$$\alpha_{m+1} = \frac{\alpha\alpha_m}{\sqrt{m+1}}$$

implies that:

$\underline{\underline{m=0}}$ $\quad \alpha_1 = \frac{\alpha\alpha_0}{\sqrt{0+1}} = \alpha\alpha_0$

$\underline{\underline{m=1}}$ $\quad \alpha_2 = \frac{\alpha\alpha_1}{\sqrt{1+1}} = \frac{\alpha}{\sqrt{2}}\alpha\alpha_0 = \frac{\alpha^2}{\sqrt{2}}\alpha_0$

$\underline{\underline{m=2}}$ $\quad \alpha_3 = \frac{\alpha\alpha_2}{\sqrt{2+1}} = \frac{\alpha}{\sqrt{3}}\cdot\frac{\alpha^2}{\sqrt{2}}\alpha_0 = \frac{\alpha^3\alpha_0}{\sqrt{3!}}$

$\underline{\underline{m=3}}$ $\quad \alpha_4 = \frac{\alpha\alpha_3}{\sqrt{3+1}} = \frac{\alpha}{\sqrt{4}}\cdot\frac{\alpha^3\alpha_0}{\sqrt{3!}} = \frac{\alpha^4}{\sqrt{4!}}\alpha_0$

...

$$\Rightarrow \alpha_m = \frac{\alpha^m}{\sqrt{n!}}\alpha_0$$

and so our coherent states are: $|\alpha\rangle = \sum_{m=0}^{\infty}\frac{\alpha^m\alpha_0}{\sqrt{n!}}|m\rangle$

$$= \alpha_0\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{n!}}|m\rangle.$$

We fix α_0 by demanding that the coherent states be normalised:

$\langle\alpha|\alpha\rangle = 1$, by demand.

$$\underbrace{\left(\alpha_0^*\sum_{m=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\langle m|\right)}_{\langle\alpha|}\left(\alpha_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right) = 1$$

$$\Rightarrow \|\alpha_0\|^2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\underbrace{\langle m|n\rangle}_{\delta_{mn}} = 1$$

$$\|\alpha_0\|^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m} \alpha^n}{\sqrt{m!n!}} \delta_{mn} = 1$$

$$\|\alpha_0\|^2 \sum_{m=0}^{\infty} \frac{\alpha^{*m} \alpha^m}{\sqrt{m!m!}} = 1$$

$$\|\alpha_0\|^2 \underbrace{\sum_{m=0}^{\infty} \frac{(\|\alpha\|^2)^m}{m!}}_{\text{series expansion for } e^{\|\alpha\|^2}} = 1$$

$$\|\alpha_0\|^2 e^{\|\alpha\|^2} = 1$$

$$\|\alpha_0\| = e^{-\frac{1}{2}\|\alpha\|^2}$$

$$\alpha_0 = e^{-\frac{1}{2}\|\alpha\|^2} e^{i\phi}$$

phase factor

set $\phi = 0$

$$\alpha_0 = e^{-\frac{1}{2}\|\alpha\|^2}$$

$$\Rightarrow |\alpha\rangle = e^{-\frac{1}{2}\|\alpha\|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{n!}} |m\rangle$$

Q.E.D.

(c) we have already answered this, in (b).

Exercise #43

$$\langle\alpha|\hat{\cos}\phi|\alpha\rangle = \left(e^{-\frac{1}{2}\|\alpha\|^2} \sum_{m=0}^{\infty} \frac{\alpha^{*m}}{\sqrt{m!}} \langle m|\right) \hat{\cos}\phi \left(e^{-\frac{1}{2}\|\alpha\|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\right) \quad \text{by (268)}$$

$$= e^{-\|\alpha\|^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} \langle m|\hat{\cos}\phi|n\rangle$$

$$= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} \langle m|e^{i\hat{\phi}} + e^{-i\hat{\phi}}|n\rangle \quad \text{by (257a)}$$

(a)
$$= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} \left(\langle m|\frac{1}{\sqrt{\hat{n}+1}}\hat{a}|n\rangle + \langle m|\hat{a}^{\dagger}\frac{1}{\sqrt{\hat{n}+1}}|n\rangle\right) \quad \text{by (252)}$$

Now, work separately on the two terms in brackets.

(b) $\langle m|\frac{1}{\sqrt{\hat{n}+1}}\hat{a}|n\rangle = \begin{cases} 0, & n=0 \\ \langle m|n-1\rangle, & n\neq 0 \end{cases} = (1-\delta_{n0})\delta_{m,n-1}$

(c) $\langle m|\hat{a}^{\dagger}\frac{1}{\sqrt{\hat{n}+1}}|n\rangle = \frac{1}{\sqrt{n+1}}\langle m|\hat{a}^{\dagger}|n\rangle = \frac{1}{\cancel{\sqrt{n+1}}}\langle m|\cancel{\sqrt{n+1}}|n+1\rangle = \delta_{m,n+1}$

using (b) and (c), (a) becomes:

$$\langle\alpha|\hat{\cos}\phi|\alpha\rangle = \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} \left((1-\delta_{n0})\delta_{m,n-1} + \delta_{m,n+1}\right)$$

(d)
$$= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} (1-\delta_{n0})\delta_{m,n-1} + \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}} \delta_{m,n+1}$$

We separately consider the two terms on the right side of (d).

First term $= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!\,n!}} \delta_{m,n-1}$ (only terms with $m=n-1$ are kept, i.e. $n=m+1$.)

note

$$= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \frac{\alpha^{*m}\alpha^{m+1}}{\sqrt{m!\,(m+1)!}} \quad (e)$$

Second term $= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!\,n!}} \delta_{m,n+1} = \frac{e^{-\|\alpha\|^2}}{2} \sum_{n=0}^{\infty} \frac{\alpha^{*n+1}\alpha^n}{\sqrt{(n+1)!\,n!}}$

$$= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \frac{\alpha^{*m+1}\alpha^m}{\sqrt{(m+1)!\,m!}} \quad (f)$$

Now put (e) and (f) into (d):

$$\Rightarrow \langle\alpha|\hat{\cos}\,\phi|\alpha\rangle = \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \frac{\alpha^{*m}\alpha^{m+1} + \alpha^{*m+1}\alpha^m}{\sqrt{m!\,(m+1)!}} \text{ as required.} \quad (g)$$

To prove the second part of equation (277) in the lecture notes, recall from (276) that: $\alpha = \|\alpha\| e^{i\theta}$. Hence (g) becomes:

$$\langle\alpha|\hat{\cos}\,\phi|\alpha\rangle = \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \frac{\alpha^{*m}\alpha^m(\alpha+\alpha^*)}{\sqrt{m!\,(m+1)!}}$$

$$= \frac{e^{-\|\alpha\|^2}}{2} \sum_{m=0}^{\infty} \frac{(\alpha^*\alpha)^m \times 2\,\mathrm{Re}(\alpha)}{\sqrt{m!\,(m+1)!}}$$

$$= e^{-\|\alpha\|^2} \sum_{m=0}^{\infty} \frac{\|\alpha\|^{2m}\,\mathrm{Re}\,\|\alpha\| e^{i\theta}}{\sqrt{m!\,(m+1)!}}$$

$$= \|\alpha\| e^{-\|\alpha\|^2} \cos\theta \sum_{m=0}^{\infty} \frac{\|\alpha\|^{2m}}{\sqrt{m!\,(m+1)!}}$$

$$= \|\alpha\| e^{-\|\alpha\|^2} \cos\theta \sum_{m=0}^{\infty} \frac{\|\alpha\|^{2m}}{\sqrt{m!\,(m+1)m!}} = \|\alpha\| e^{-\|\alpha\|^2} \cos\theta \sum_{m=0}^{\infty} \frac{\|\alpha\|^{2m}}{m!\sqrt{m+1}}$$

Q.E.D.

Exercise #44 $\hat{\cos}^2\phi = \frac{1}{2}(e^{i\hat\phi}+e^{-i\hat\phi})\frac{1}{2}(e^{i\hat\phi}+e^{-i\hat\phi})$

$$= \frac{1}{4}\left((e^{i\hat\phi})^2 + e^{i\hat\phi}e^{-i\hat\phi} + e^{-i\hat\phi}e^{i\hat\phi} + (e^{-i\hat\phi})^2\right)$$

Hence $\langle\alpha|\hat{\cos}^2\phi|\alpha\rangle = \frac{1}{4}\{\underbrace{\langle\alpha|(e^{i\hat\phi})^2|\alpha\rangle}_{(a)} + \underbrace{\langle\alpha|e^{i\hat\phi}e^{-i\hat\phi}|\alpha\rangle}_{(b)} + \underbrace{\langle\alpha|e^{-i\hat\phi}e^{i\hat\phi}|\alpha\rangle}_{(c)} + \underbrace{\langle\alpha|(e^{-i\hat\phi})^2|\alpha\rangle}_{(d)}$

(α)

we separately work out pieces ⓐ, ⓑ, ⓒ, ⓓ.

Piece ⓐ Making use of the Fock expansion (268) of the coherent states, we can kick off the derivation:

$$\langle\alpha|(e^{i\hat\phi})^2|\alpha\rangle = e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|e^{i\hat\phi}\underbrace{e^{i\hat\phi}|n\rangle}_{\xrightarrow{(253a)}\begin{cases}|n-1\rangle, & \text{if } n\neq 0\\ 0, & \text{if } n=0\end{cases}}$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|e^{i\hat\phi}|n-1\rangle$$

note

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\underbrace{\langle m|n-2\rangle}_{\delta_{m,n-2}} \quad \text{... we used (253 a) again!}$$

note

$$\Rightarrow m = n-2 \qquad \Rightarrow n = m+2$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\alpha^{*m}\alpha^{m+2}}{\sqrt{m!(m+2)!}}$$

Piece ⓑ

$$\langle\alpha|e^{i\hat\phi}e^{-i\hat\phi}|\alpha\rangle = e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|e^{i\hat\phi}\underbrace{e^{-i\hat\phi}|n\rangle}_{\to |n+1\rangle, \text{ by } (253b)}$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|\underbrace{e^{i\hat\phi}|n+1\rangle}_{\to |n\rangle, \text{ by } (253a)}$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\underbrace{\langle m|n\rangle}_{\to\delta_{mn}}$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\alpha^{*m}\alpha^m}{m!} = e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{(\|\alpha\|^2)^m}{m!} = e^{-\|\alpha\|^2}e^{\|\alpha\|^2} = \underline{1}$$

Piece ⓒ $\langle\alpha|e^{-i\hat\phi}e^{i\hat\phi}|\alpha\rangle$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|e^{-i\hat\phi}\underbrace{e^{i\hat\phi}|n\rangle}_{\xrightarrow{(253a)}\begin{cases}|n-1\rangle, & \text{if } n\neq 0\\ 0, & \text{if } n=0\end{cases}}$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|\underbrace{e^{-i\hat\phi}|n-1\rangle}_{\to \text{By } (253b),\ e^{-i\hat\phi}|n-1\rangle = |n\rangle}$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\underbrace{\langle m|n\rangle}_{\to\delta_{mn}}$$

$$= e^{-\|\alpha\|^2}\sum_{m=1}^{\infty}\frac{\|\alpha\|^{2m}}{m!}$$

note!!

$$= e^{-\|\alpha\|^2}\Bigg(\underbrace{\sum_{m=0}^{\infty}\frac{\|\alpha\|^{2m}}{m!}}_{e^{\|\alpha\|^2}} - \underbrace{\frac{\|\alpha\|^{2\times 0}}{0!}}_{1}\Bigg)$$

$$= e^{-\|\alpha\|^2}\left(e^{\|\alpha\|^2}-1\right) = 1-e^{-\|\alpha\|^2}$$

Piece (d)

$$\langle\alpha|(e^{-i\hat{\phi}})^2|\alpha\rangle = e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\langle m|e^{-i\hat{\phi}}\underbrace{e^{-i\hat{\phi}}|n\rangle}_{|n+2\rangle,\text{ by (253b)}}$$

$$= e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\underbrace{\langle m|n+2\rangle}_{\delta_{m,n+2}}$$

$m=n+2$

$$= e^{-\|\alpha\|^2}\sum_{n=0}^{\infty}\frac{\alpha^{*(n+2)}\alpha^n}{\sqrt{(n+2)!n!}}$$

Now collect together pieces (a), (b), (c), (d) and substitute into equation (α):

$$\Rightarrow \langle\alpha|\hat{\cos}^2\phi|\alpha\rangle = \frac{1}{4}\left((a)+(b)+(c)+(d)\right)$$

$$= \frac{1}{4}\left[e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\alpha^{*m}\alpha^{m+2}}{\sqrt{m!(m+2)!}} + 1 + 1 - e^{-\|\alpha\|^2} + e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\alpha^{*m+2}\alpha^m}{\sqrt{m!(m+2)!}}\right]$$

$$= \frac{1}{4}\left[2 - e^{-\|\alpha\|^2} + e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\alpha^{*m}\alpha^{m+2}+\alpha^{*m+2}\alpha^m}{\sqrt{m!(m+2)!}}\right]$$

$$= \frac{1}{4}\left[2 - e^{-\|\alpha\|^2} + e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\|\alpha\|^{2m}(\alpha^2+\alpha^{*2})}{\sqrt{m!(m+2)!}}\right] \qquad (\beta)$$

Now, $\alpha^2+\alpha^{*2} = (\|\alpha\|e^{i\theta})^2 + (\|\alpha\|e^{-i\theta})^2$

$$= \|\alpha\|^2(e^{2i\theta}+e^{-i2\theta})$$

$$= 2\|\alpha\|^2\cos 2\theta$$

$$= 2\|\alpha\|^2(2\cos^2\theta - 1)$$

$$= 4\|\alpha\|^2(\cos^2\theta - \tfrac{1}{2})$$

Hence (β) becomes:

$$\langle\alpha|\hat{\cos}^2\phi|\alpha\rangle = \frac{1}{4}\left[2 - e^{-\|\alpha\|^2} + e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\|\alpha\|^{2m}\times 4\|\alpha\|^2(\cos^2\theta-\frac{1}{2})}{\sqrt{m!(m+2)!}}\right]$$

$$= \frac{1}{2} - \frac{1}{4}e^{-\|\alpha\|^2} + \|\alpha\|^2(\cos^2\theta - \tfrac{1}{2})e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\|\alpha\|^{2m}}{\sqrt{m!(m+2)(m+1)m!}}$$

$$= \frac{1}{2} - \frac{1}{4}e^{-\|\alpha\|^2} + \|\alpha\|^2(\cos^2\theta - \tfrac{1}{2})e^{-\|\alpha\|^2}\sum_{m=0}^{\infty}\frac{\|\alpha\|^{2m}}{m!\sqrt{(m+1)(m+2)}} \quad \text{Q.E.D.}$$

Exercise #45

$$\langle\alpha|\hat{E}^2|\alpha\rangle \overset{(238a)}{=} -\frac{\hbar\omega}{2V}\langle\alpha|\left(\hat{a}e^{i(\vec{k}\cdot\vec{x}-\omega t)} - \hat{a}^{+}e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\right)^2|\alpha\rangle$$

$$= -\frac{\hbar\omega}{2V}\langle\alpha|\hat{a}^2e^{2i(\vec{k}\cdot\vec{x}-\omega t)} + \hat{a}^{+2}e^{-2i(\vec{k}\cdot\vec{x}-\omega t)} - \hat{a}^{+}\hat{a} - \underbrace{\hat{a}\hat{a}^{+}}_{=\hat{a}^{+}\hat{a}+1,\text{ by (203)}}|\alpha\rangle$$

$$= -\frac{\hbar\omega}{2V}\langle\alpha|\hat{a}^2 e^{2i(\vec{k}\cdot\vec{x}-\omega t)} + \hat{a}^{+2} e^{-2i(\vec{k}\cdot\vec{x}-\omega t)} - 2\hat{a}^+\hat{a} - 1|\alpha\rangle$$

(now use $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\hat{a}^+ = \alpha^*\langle\alpha|$)

$$= -\frac{\hbar\omega}{2V}\langle\alpha|\alpha^2 e^{2i(\vec{k}\cdot\vec{x}-\omega t)} + \alpha^{*2} e^{-2i(\vec{k}\cdot\vec{x}-\omega t)} - 2\alpha^*\alpha - 1|\alpha\rangle$$

$$= -\frac{\hbar\omega}{2V}\left(\alpha^2 e^{2i(\vec{k}\cdot\vec{x}-\omega t)} + \alpha^{*2} e^{-2i(\vec{k}\cdot\vec{x}-\omega t)} - 2\|\alpha\|^2 - 1\right)\underbrace{\langle\alpha|\alpha\rangle}_{\to 1, \text{ by } (269)}$$

(now let $\|\alpha\| e^{i\theta} = \alpha$)

$$= -\frac{\hbar\omega}{2V}\left(\|\alpha\|^2 e^{2i\theta} e^{2i(\vec{k}\cdot\vec{x}-\omega t)} + \|\alpha\|^2 e^{-2i\theta} e^{-2i(\vec{k}\cdot\vec{x}-\omega t)} - 2\|\alpha\|^2 - 1\right)$$

$$= -\frac{\hbar\omega}{2V}\left(\|\alpha\|^2\left\{\underbrace{e^{2i(\vec{k}\cdot\vec{x}-\omega t+\theta)} + e^{-2i(\vec{k}\cdot\vec{x}-\omega t+\theta)}}_{2\cos(2(\vec{k}\cdot\vec{x}-\omega t+\theta))} - 2\right\} - 1\right)$$

$$= -\frac{\hbar\omega}{2V}\left(2\|\alpha\|^2\left\{\cos(2(\vec{k}\cdot\vec{x}-\omega t+\theta)) - 1\right\} - 1\right)$$

(we now make use of a double-angle formula from trigonometry:
$\cos 2\beta - 1 = -2\sin^2\beta$)

$$= -\frac{\hbar\omega}{2V}\left(2\|\alpha\|^2\left\{-2\sin^2(\vec{k}\cdot\vec{x}-\omega t+\theta)\right\} - 1\right)$$

$$= -\frac{\hbar\omega}{2V}\left(-4\|\alpha\|^2\sin^2(\vec{k}\cdot\vec{x}-\omega t+\theta) - 1\right)$$

$$= \frac{\hbar\omega}{2V}\left(4\|\alpha\|^2\sin^2(\vec{k}\cdot\vec{x}-\omega t+\theta) + 1\right) \quad \text{Q.E.D.}$$

April 9, 2002

Exercise #46 Start with the Fock expansion (268) of the coherent states: $|\alpha\rangle = \exp(-\frac{1}{2}\|\alpha\|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$.

Our next task is to write the number/Fock states $|n\rangle$ in terms of a certain operator acting on the vacuum $|0\rangle$; this was done in exercise #35, leading to equation (219) of the lectures, namely: $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$. Hence the Fock expansion of the coherent states becomes: $|\alpha\rangle = e^{-\frac{1}{2}\|\alpha\|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$

$$= e^{-\frac{1}{2}\|\alpha\|^2}\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^+)^n}{n!}|0\rangle \quad \text{Q.E.D.} \quad (a)$$

Continuing further, note that $\sum_{n=0}^{\infty} \frac{(\alpha \hat{a}^{+})^n}{n!} = e^{\alpha \hat{a}^{+}}$, and so:

$$|\alpha\rangle = e^{-\frac{1}{2}\|\alpha\|^2} e^{\alpha \hat{a}^{+}} |0\rangle \quad (b)$$

Now let $\hat{c} \equiv -\frac{1}{2}\|\alpha\|^2$ and $\hat{d} \equiv \alpha \hat{a}^{+}$. Clearly $[\hat{c}, \hat{d}] = 0$, because $\hat{c}$ is just a number. Therefore condition (289) holds and so:

$$|\alpha\rangle = e^{-\frac{1}{2}\|\alpha\|^2} e^{\alpha \hat{a}^{+}} |0\rangle$$

$$= \exp\left(-\tfrac{1}{2}\|\alpha\|^2 + \alpha \hat{a}^{+} + \tfrac{1}{2}\underbrace{\cancel{[-\tfrac{1}{2}\|\alpha\|^2, \alpha \hat{a}^{+}]}}_{\to 0}\right)|0\rangle \qquad (\text{by } (290))$$

$$= \exp\left(-\tfrac{1}{2}\|\alpha\|^2 + \alpha \hat{a}^{+}\right)|0\rangle$$

$$= \exp\left(\alpha \hat{a}^{+} - \tfrac{1}{2}\|\alpha\|^2\right)|0\rangle \quad \text{Q.E.D.} \qquad (c)$$

To complete the exercise, we need to prove equation (291) from the lectures. The proof begins with a trick, which makes use of the fact that $\hat{a}|0\rangle = 0$:

$$e^{-\alpha^* \hat{a}}|0\rangle = [1 + \underbrace{(-\alpha^* \hat{a}) + \frac{(-\alpha^* \hat{a})^2}{2!} + \dots}_{\text{all give 0 when acting on } |0\rangle}]|0\rangle = |0\rangle \qquad (d)$$

Hence (c) can be written as:

$$|\alpha\rangle = \exp\left(\alpha \hat{a}^{+} - \tfrac{1}{2}\|\alpha\|^2\right) \exp\left(-\alpha^* \hat{a}\right)|0\rangle \qquad (e)$$

Now let $\hat{c} \equiv \alpha \hat{a}^{+} - \frac{1}{2}\|\alpha\|^2$ and $\hat{d} \equiv -\alpha^* \hat{a}$. It is easy to show that these satisfy (289) of the lecture notes, which allows us to use relation (290) to rewrite (e) as:

$$(f) \quad |\alpha\rangle = \exp\left(\alpha \hat{a}^{+} - \tfrac{1}{2}\|\alpha\|^2 - \alpha^* \hat{a} + \tfrac{1}{2}[\alpha \hat{a}^{+} - \tfrac{1}{2}\|\alpha\|^2, -\alpha^* \hat{a}]\right)|0\rangle$$

Now, (g) $[\alpha \hat{a}^{+} - \frac{1}{2}\|\alpha\|^2, -\alpha^* \hat{a}] = [\alpha \hat{a}^{+}, -\alpha^* \hat{a}] = -\|\alpha\|^2 \underbrace{[\hat{a}^{+}, \hat{a}]}_{-1,\ \text{by } (203)} = +\|\alpha\|^2$

and so (f) becomes:

$$(h) \quad |\alpha\rangle = \exp\left(\alpha \hat{a}^{+} \cancel{- \tfrac{1}{2}\|\alpha\|^2} - \alpha^* \hat{a} \cancel{+ \tfrac{1}{2}(+\|\alpha\|^2)}\right)|0\rangle$$

$$|\alpha\rangle = \exp\left(\alpha \hat{a}^{+} - \alpha^* \hat{a}\right)|0\rangle \quad \text{Q.E.D.}$$

Exercise #47

$$\frac{1}{\pi}\iint |\alpha\rangle\langle\alpha|\, d^2\alpha = \frac{1}{\pi}\iint \left(e^{-\frac{1}{2}\|\alpha\|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}}|m\rangle\right)\left(e^{-\frac{1}{2}\|\alpha\|^2} \sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}}\langle n|\right) d^2\alpha$$

$$= \frac{1}{\pi} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|m\rangle\langle n|}{\sqrt{m!\,n!}} \iint \alpha^m \alpha^{*n}\, d^2\alpha \times e^{-\|\alpha\|^2}$$

(Now let $\alpha = re^{i\theta}$)

$$= \frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|m\rangle\langle n|}{\sqrt{m!n!}} \underbrace{\int_0^{2\pi} d\theta \int_0^{\infty} r\,dr}_{\text{plane polar coordinates}} \overbrace{r^m e^{im\theta}}^{\alpha^m} \overbrace{r^n e^{-in\theta}}^{\alpha^{*n}} e^{\overbrace{-r^2}^{e^{-|\alpha|^2}}}$$

$$= \frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|m\rangle\langle n|}{\sqrt{m!n!}} \underbrace{\int_0^{2\pi} e^{i(m-n)\theta} d\theta}_{2\pi\delta_{mn}} \int_0^{\infty} r^{m+n+1} e^{-r^2} dr$$

$$= \cancel{\frac{1}{\pi}} \sum_{m=0}^{\infty} \frac{|m\rangle\langle m|}{m!} \times 2\cancel{\pi} \times \underbrace{\int_0^{\infty} r^{2m+1} e^{-r^2} dr}$$ By (29.3), this is: $m!$

$$= \sum_{m=0}^{\infty} \frac{|m\rangle\langle m|}{\cancel{m!}} \cancel{m!} = \sum_{m=0}^{\infty} |m\rangle\langle m| = 1$$

Exercise #48 (a) Expectation value of photon number:

$$\langle\theta|\hat{n}|\theta\rangle = \left(\frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} e^{-im\theta} \langle m|\right) \hat{a}^{+}\hat{a} \left(\frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta} |n\rangle\right)$$

$$= \frac{1}{s+1} \sum_{m=0}^{s} \sum_{n=0}^{s} e^{i(n-m)\theta} \langle m|\underbrace{\hat{a}^{+}\hat{a}|n\rangle}_{\to\, n|n\rangle}$$

$$= \frac{1}{s+1} \sum_{m=0}^{s} \sum_{n=0}^{\infty} e^{i(n-m)\theta} \cdot n \underbrace{\langle m|n\rangle}_{\to\,\delta_{mn}}$$

$$= \frac{1}{s+1} \sum_{m=0}^{s} m = \frac{1}{s+1} \times \frac{1}{2} s(s+1) = \frac{1}{2} s$$

(b) $$\langle\theta|\widehat{\cos\phi}|\theta\rangle = \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} e^{-im\theta} \langle m| \frac{1}{2}(e^{i\hat{\phi}} + e^{-i\hat{\phi}}) \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta} |n\rangle$$

$$= \frac{1}{2(s+1)} \sum_{m=0}^{s} \sum_{n=0}^{s} e^{i(n-m)\theta} \langle m| e^{i\hat{\phi}} + e^{-i\hat{\phi}} |n\rangle$$

$$= \frac{1}{2(s+1)} \left\{ \sum_{m=0}^{s} \sum_{n=0}^{s} e^{i(n-m)\theta} \underbrace{\langle m| e^{i\hat{\phi}}|n\rangle}_{\text{use } (253a)} + \sum_{m=0}^{s} \sum_{n=0}^{s} e^{i(n-m)\theta} \underbrace{\langle m| e^{-i\hat{\phi}}|n\rangle}_{\text{use } (253b)} \right\}$$

$$= \frac{1}{2(s+1)} \left\{ \sum_{m=0}^{s} \sum_{n=1}^{s} e^{i(n-m)\theta} \underbrace{\langle m|n-1\rangle}_{\delta_{m,n-1}} + \sum_{m=0}^{s} \sum_{n=0}^{s} e^{i(n-m)\theta} \underbrace{\langle m|n+1\rangle}_{\delta_{m,n+1}} \right\}$$

note!! ($n=1$)

$m=n-1 \Rightarrow n=m+1$ $\qquad$ $m=n+1$

$$= \frac{1}{2(s+1)} \left\{ \sum_{m=0}^{s} e^{i(m+1-m)\theta} + \sum_{n=0}^{s} e^{i(n-(n+1))\theta} \right\}$$

$$= \frac{1}{2(s+1)} \sum_{m=0}^{s} (e^{i\theta} + e^{-i\theta}) = \frac{1}{2(s+1)} \cdot 2\cos\theta \cdot (s+1) = \cos\theta$$

A

Exercise #51. We first show that our multimode coherent state, defined by:

$$(1)\quad \hat{a}_{\vec{k}_i} | \alpha_{\vec{k}_1} \alpha_{\vec{k}_2} \cdots \alpha_{\vec{k}_i} \cdots \rangle = \alpha_{\vec{k}_i} | \alpha_{\vec{k}_1} \alpha_{\vec{k}_2} \cdots \alpha_{\vec{k}_i} \cdots \rangle,$$

are also eigenfunctions of the $\hat{E}^+$ operator defined in (315 a) of the notes.

$$(2)\quad \hat{E}^+(\vec{r},t) | \alpha_{\vec{k}_1} \alpha_{\vec{k}_2} \cdots \rangle \quad \left(\text{now use } (315a)\right)$$

$$= i \sum_{\vec{k}} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\, \hat{a}_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} | \alpha_{\vec{k}_1} \alpha_{\vec{k}_2} \cdots \rangle$$

(note that the $\vec{\varepsilon}_{\vec{k}}$ has been dropped)

$$= i \sum_{\vec{k}} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\, \alpha_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} | \alpha_{\vec{k}_1} \alpha_{\vec{k}_2} \cdots \rangle$$

$$\equiv \mathcal{E}(\vec{r},t) | \alpha_{\vec{k}_1} \alpha_{\vec{k}_2} \cdots \rangle$$

where, by definition, the eigenvalue $\mathcal{E}$ of $\hat{E}^+$ is:

$$(3)\quad \mathcal{E}(\vec{r},t) = i \sum_{\vec{k}} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\, \alpha_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}.$$

We are now ready to calculate the quantum correlation function $g^{(1)}$ defined in equation (330) of the notes:

B

7) $g^{(r)}(\vec{r}_1, t_1, \ldots, \vec{r}_r, t_r; \vec{r}_{r+1}, t_{r+1}, \ldots, \vec{r}_{2r}, t_{2r})$

$$= \frac{\left\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \left| \prod_{i=1}^{r} \hat{E}^-(\vec{r}_i, t_i) \prod_{j=r+1}^{2r} \hat{E}^+(\vec{r}_j, t_j) \right| \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \right\rangle}{\prod_{i=1}^{2r} \sqrt{\left\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \left| \hat{E}^-(\vec{r}_i, t_i)\hat{E}^+(\vec{r}_i, t_i) \right| \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \right\rangle}}$$

(now make use of ② and its Hermitian adjoint, $\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots | \hat{E}^- = \langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots | \mathcal{E}^*(\vec{r}, t)$)

$$= \frac{\left\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \left| \prod_{i=1}^{r} \mathcal{E}^*(\vec{r}_i, t_i) \prod_{j=r+1}^{2r} \mathcal{E}(\vec{r}_j, t_j) \right| \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \right\rangle}{\prod_{i=1}^{2r} \sqrt{\left\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \left| \mathcal{E}^*(\vec{r}_i, t_i)\mathcal{E}(\vec{r}_i, t_i) \right| \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \right\rangle}}$$

$$= \frac{\prod_{i=1}^{r} \mathcal{E}^*(\vec{r}_i, t_i) \prod_{j=r+1}^{2r} \mathcal{E}(\vec{r}_j, t_j) \left\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots | \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \right\rangle}{\prod_{i=1}^{2r} \sqrt{\|\mathcal{E}(\vec{r}_i, t_i)\|^2} \sqrt{\left\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots | \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \right\rangle}}$$

($\langle \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots | \alpha_{\vec{k}_1}\alpha_{\vec{k}_2}\cdots \rangle = 1$ because our multimode coherent states are assumed to be normalised.)

$$\boxed{6}$$

$$= \frac{\prod_{i=1}^{r} \mathcal{E}^*(\vec{r}_i, t_i) \prod_{j=r+1}^{2r} \mathcal{E}(\vec{r}_j, t_j)}{\prod_{i=1}^{2r} \left(\| \mathcal{E}(\vec{r}_i, t_i) \| \right)}$$

(now let $\mathcal{E}(\vec{r}, t) = \|\mathcal{E}(\vec{r}, t)\| e^{i\theta(\vec{r}, t)}, \theta \in \mathbb{R}$)

$$= \frac{\prod_{i=1}^{r} \|\mathcal{E}(\vec{r}_i, t_i)\| e^{-i\theta(\vec{r}_i, t_i)} \prod_{j=r+1}^{2r} \|\mathcal{E}(\vec{r}_j, t_j)\| e^{i\theta(\vec{r}_j, t_j)}}{\prod_{i=1}^{2r} \left(\| \mathcal{E}(\vec{r}_i, t_i) \| \right)}$$

$$= \prod_{i=1}^{r} e^{-i\theta(\vec{r}_i, t_i)} \prod_{j=r+1}^{2r} e^{i\theta(\vec{r}_j, t_j)}$$

Hence,

5 $$\| g^{(r)}(\vec{r}_1, t_1, \ldots, \vec{r}_r, t_r ; \vec{r}_{r+1}, t_{r+1}, \ldots, \vec{r}_{2r}, t_{2r}) \|$$

$$= \left\| \prod_{i=1}^{r} e^{-i\theta(\vec{r}_i, t_i)} \prod_{j=r+1}^{2r} e^{i\theta(\vec{r}_j, t_j)} \right\|$$

$$= 1.$$

Therefore the multimode coherent states are coherent to all orders.

3/4/2009

Y

Quantum mechanics – Problems #52–60, Solutions David Paganin

Problem 52 (a) Take the curl of (178b) and then make use of the vector identity $\nabla\times\nabla\times = \nabla\nabla\cdot - \nabla^2$, to give

$$\underbrace{\nabla\nabla\cdot\vec{B}}_{0,\text{ by (178c)}} - \nabla^2\vec{B} = \underbrace{c^{-1}\nabla\times\vec{j}}_{\vec{0},\text{ in absence of currents}} + c^{-1}\partial_t\underbrace{\nabla\times\vec{E}}_{-c^{-1}\partial_t\vec{B},\text{ by (178d)}}$$

$$\Rightarrow -\nabla^2\vec{B} = c^{-1}\partial_t(-c^{-1}\partial_t\vec{B})$$

$$\Rightarrow (c^{-2}\partial_t^2 - \nabla^2)\vec{B} = 0$$

Let $\vec{B} \equiv \vec{B}(\vec{x},t) = \vec{B}(\vec{x})e^{-i\omega t}$

$$\Rightarrow (c^{-2}\partial_t^2 - \nabla^2)\vec{B}(\vec{x})e^{-i\omega t} = 0$$

$$\Rightarrow (c^{-2}(-i\omega)^2 - \nabla^2)\vec{B}(\vec{x})e^{-i\omega t} = 0$$

$$\Rightarrow (\nabla^2 + \underbrace{c^{-2}\omega^2}_{=k^2})\vec{B}(\vec{x}) = 0$$

Therefore each component of $\vec{B}(\vec{x})$ is a Helmholtz field. Q.E.D.

Similarly, take the curl of (178d), and make use of the previously mentioned vector identity, to give:

$$\nabla\nabla\cdot\vec{E} - \nabla^2\vec{E} = -c^{-1}\partial_t\,\nabla\times\vec{B}$$

$\nabla\cdot\vec{E} = \rho$, by (178a) $= 0$, since we are in free space

$\nabla\times\vec{B} = c^{-1}\vec{j} + c^{-1}\partial_t\vec{E}$, by (178b) $= c^{-1}\partial_t\vec{E}$, since there are no currents

$$\Rightarrow \nabla(0) - \nabla^2\vec{E} = -c^{-1}\partial_t(c^{-1}\partial_t\vec{E})$$

$\Rightarrow (c^{-2}\partial_t^2 - \nabla^2)\vec{E} = 0$. Identical to the previously-derived equation for $\vec{B}$, which was already shown to imply that $\vec{B}(\vec{x},t)$, and hence $\vec{E}(\vec{x},t)$, is a Helmholtz field.

(b) Trivial.

(c) The free space Klein-Gordon equation is given in (12).

$$\Rightarrow (c^{-2}\partial_t^2 - \nabla^2 + m^2)\Psi(\vec{x},t) = 0$$

note

$$\Rightarrow (c^{-2}\partial_t^2 - \nabla^2 + m^2)\Psi(\vec{x})e^{-i\omega t} = 0$$

$$\Rightarrow (c^{-2}(-i\omega)^2 - \nabla^2 + m^2)\Psi(\vec{x})\cancel{e^{-i\omega t}} = 0$$

$$\Rightarrow (\nabla^2 + c^{-2}\omega^2 - m^2)\Psi(\vec{x}) = 0 \quad \text{Q.E.D.}$$

(d) (53) $\Rightarrow (E - \vec{\alpha}\cdot\vec{p} - \beta m)\Psi(\vec{x},t) = 0$ → 4-component spinor

Apply $(E + \vec{\alpha}\cdot\vec{p} + \beta m)$ to both sides:

$$\Rightarrow (E+\vec{\alpha}\cdot\vec{p}+\beta m)(E-\vec{\alpha}\cdot\vec{p}-\beta m)\Psi(\vec{x},t)=0$$

$$(E^2 - \cancel{E\vec{\alpha}\cdot\vec{p}} - \cancel{E\beta m} + \cancel{\vec{\alpha}\cdot\vec{p}E} - (\vec{\alpha}\cdot\vec{p})^2 - \vec{\alpha}\cdot\vec{p}\beta m$$

$$+ \cancel{\beta m E} - \beta m\vec{\alpha}\cdot\vec{p} - \beta^2 m^2)\Psi(\vec{x},t) = 0$$

$\beta = 1$, by (54b)

$$(E^2 - (\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)^2 - \underbrace{[\vec{\alpha}\cdot\vec{p}, \beta m]_+}_{\to 0} - m^2)\Psi = 0$$

$\to 0$, since $\vec{\alpha}$ and β anticommute (see 54d)

NOW multiply by -1 ...

$$(E^2 + \underbrace{\alpha_x^2}_{=1\ (\text{see }54a)} p_x^2 + \alpha_x\alpha_y p_x p_y + \alpha_x\alpha_z p_x p_z + \underbrace{\alpha_y\alpha_x}_{=-\alpha_x\alpha_y,\ \text{by }(54c)} p_y p_x$$

$$+ \underbrace{\alpha_y^2}_{=1\ (\text{see }54a)} p_y^2 + \alpha_y\alpha_z p_y p_z + \underbrace{\alpha_z\alpha_x}_{\to -\alpha_x\alpha_z} p_z p_x + \underbrace{\alpha_z\alpha_y}_{\to -\alpha_y\alpha_z} p_z p_y$$

$$+ \underbrace{\alpha_z^2}_{=1\ (\text{see }54a)} p_z^2 + m^2)\Psi(\vec{x},t) = 0$$

$$(-E^2 + \underbrace{p_x^2 + p_y^2 + p_z^2}\ \boxed{+\alpha_x\alpha_y p_x p_y - \alpha_x\alpha_y p_y p_x}\ \boxed{+\alpha_x\alpha_z p_x p_z - \alpha_x\alpha_z p_z p_x}$$

$$+ \alpha_y\alpha_z p_y p_z - \alpha_y\alpha_z p_z p_y + m^2)\Psi(\vec{x},t) = 0$$

$$(-E^2 + |\vec{p}|^2 + \alpha_x\alpha_y\underbrace{[p_x,p_y]}_{0} + \alpha_x\alpha_z\underbrace{[p_x,p_z]}_{0} + \alpha_y\alpha_z\underbrace{[p_y,p_z]}_{0} + m^2)\Psi(\vec{x},t) = 0$$

$$(-E^2 + \vec{p}^2 + m^2)\Psi(\vec{x},t) = 0$$

Let $\vec{p} = -i\nabla$, $E = i\partial_t$, $\Psi(\vec{x},t) = \Psi(x)e^{-i\omega t}$

$$\Rightarrow (\partial_t^2 - \nabla^2 + m^2)\Psi(\vec{x})e^{-i\omega t} = 0$$

$$\Rightarrow (+\omega^2 + \nabla^2 - m^2)\Psi(\vec{x}) = 0$$

Thus each component, of the free space-time independent Dirac wavefunction, is a Helmholtz field.

Problem 53 Consider equation (3.46):

$$\hat{\varphi}(k_x, k_y, z) = \hat{\varphi}(k_x, k_y, z=0)\, e^{iz\sqrt{k^2-k_x^2-k_y^2}}.$$

When $k_x^2 + k_y^2 \ll k^2$, we may evidently make the following approximation:

$$\sqrt{k^2-k_x^2-k_y^2} = k\left(1 - \frac{k_x^2+k_y^2}{k^2}\right)^{\frac{1}{2}} \approx k\left(1 - \frac{k_x^2+k_y^2}{2k^2}\right) = k - \frac{k_x^2+k_y^2}{2k},$$

so that our first equation becomes:

$$\hat{\varphi}(k_x, k_y, z) \approx \hat{\varphi}(k_x, k_y, z=0)\exp\left(iz\left(k - \frac{k_x^2+k_y^2}{2k}\right)\right)$$
$$= \exp(ikz)\,\hat{\varphi}(k_x, k_y, z=0)\exp\left(-\frac{iz}{2k}(k_x^2+k_y^2)\right).$$

While this is a bad approximation in regions where $k_x^2 + k_y^2$ is not much less than k^2, this does not matter, since in such regions the first term on the RHS of (3.46) is negligible, so the product on this right side is negligible, as it should be.

Problem 54 Elementary plane-wave solutions to the time-dependent free space Schrödinger equation are $\exp(i\vec{k}\cdot\vec{x} - i\omega t)$, where $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$. At time $t=0$, we can expand an arbitrary wavepacket as:

$$① \; \Psi(\vec{x}, t=0) = (2\pi)^{-3/2}\iiint_{-\infty}^{\infty} \hat{\Psi}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k}$$

This is a continuous sum of plane waves, "frozen" at time $t=0$. To "let" them evolve through to time t, let $\exp(i\vec{k}\cdot\vec{x}) \to \exp(i\vec{k}\cdot\vec{x} - i\omega t)$, so that:

$$② \; \Psi(\vec{x}, t) = (2\pi)^{-3/2}\iiint_{-\infty}^{\infty} \hat{\Psi}(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}\, d\vec{k}$$

Now, the inverse Fourier integral, to ①, is:

$$③ \; \hat{\Psi}(\vec{k}) = (2\pi)^{-3/2}\iiint_{-\infty}^{\infty} \Psi(\vec{x}', t=0)\, e^{-i\vec{k}\cdot\vec{x}'}\, d\vec{x}'.$$

Substitute ③ into ②. Note that we have put primes on the dummy variable $\vec{x}'$ in ③, to distinguish it from the $\vec{x}$ appearing in ②, which is not a dummy variable.

$$\Rightarrow \Psi(\vec{x},t) = (2\pi)^{-3/2}\iiint_{-\infty}^{\infty}(2\pi)^{-3/2}\iiint_{-\infty}^{\infty}\Psi(\vec{x}',t=0)\,e^{-i\vec{k}\cdot\vec{x}'}d\vec{x}' \times e^{i\vec{k}\cdot\vec{x}}\,d\vec{k}\,e^{-i\omega t}$$

$$=(2\pi)^{-3}\iiint\!\!\iiint_{-\infty}^{\infty}\Psi(\vec{x}',t=0)\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,d\vec{k}\,d\vec{x}'\,e^{-i\omega t}$$

④ $$= (2\pi)^{-3}\iiint_{-\infty}^{\infty} d\vec{x}'\left[\Psi(\vec{x}',t=0)\left\{\iiint_{-\infty}^{\infty} d\vec{k}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega t}\right\}\right]$$

Now, $\vec{p}=\hbar\vec{k}$, so that the term in braces is:

⑤ $$\iiint_{-\infty}^{\infty}\frac{d\vec{p}}{\hbar^3}\,e^{i\vec{p}\cdot(\vec{x}-\vec{x}')/\hbar}\,e^{-iEt/\hbar}$$ $\longrightarrow E=\hbar\omega \Rightarrow \omega=E/\hbar$

Hence ④ becomes:

⑥ $$\Psi(\vec{x},t) = (\underbrace{2\pi\hbar}_{h})^{-3}\iiint_{-\infty}^{\infty} d\vec{x}'\left[\Psi(\vec{x}',t=0)\left\{\iiint_{-\infty}^{\infty} d\vec{p}\,e^{i\vec{p}\cdot(\vec{x}-\vec{x}')/\hbar}\,e^{-iEt/\hbar}\right\}\right]$$

let $K(\vec{\rho};\tau) \equiv h^{-3}\iiint_{-\infty}^{\infty} d\vec{p}\,\exp\left(\frac{i}{\hbar}(\vec{p}\cdot\vec{\rho}-E\tau)\right)$

⑦ $$\Rightarrow K(\vec{x}-\vec{x}';t) = h^{-3}\iiint_{-\infty}^{\infty} d\vec{p}\,\exp\left(\frac{i}{\hbar}\left(\vec{p}\cdot(\vec{x}-\vec{x}')-Et\right)\right)$$

and so ⑥ becomes:

⑧ $$\Psi(\vec{x},t) = \iiint_{-\infty}^{\infty} d\vec{x}'\,\Psi(\vec{x}',t=0)\,K(\vec{x}-\vec{x}';t)$$

~~let~~ $\cancel{t\to t_2+t_1,\ t=0\to t=t_1 \Rightarrow \Psi(\vec{x},t_1)}$ Equation (349), in the notes, follows directly.

Problem 55 Take the real part of (374), so that:

$$\mathrm{Re}\{f(\omega_R)\} = \mathrm{Re}\left\{\frac{1}{i\pi}P\int_{-\infty}^{\infty}\frac{f(\omega)\,d\omega}{\omega-\omega_R}\right\}$$

$$=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{d\omega}{\omega-\omega_R}\mathrm{Re}\{f(\omega)/i\}$$

$$=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{d\omega}{\omega-\omega_R}\mathrm{Im}\{f(\omega)\}$$

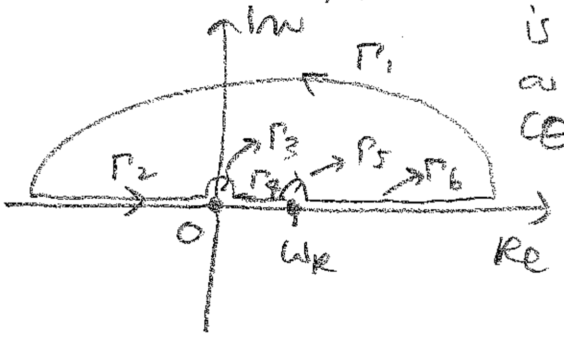
(let $\omega = a+ib$ — real numbers
$\Rightarrow \mathrm{Re}(\omega/i)$
$=\mathrm{Re}\left(\frac{a+ib}{i}\right)$
$=b$
$=\mathrm{Im}(\omega)$)

let $\omega_R\to\omega$, $\omega\to\omega'$ $\Rightarrow \mathrm{Re}\{f(\omega)\} = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega}\mathrm{Im}\{f(\omega')\}$.

Q.E.D. The lower member, of (375), is proved similarly, by taking the imaginary part of (374).

Problem 56 The required generalization of (370) is:

(1) $$\oint_\Gamma \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} = 0,$$ where the closed contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_6$ is as shown in the figure. Γ_1 is a semicircle of radius R centred at $\omega = \omega_R$, with $R \to \infty$; Γ_3 and Γ_5 are semicircles of radius $\varepsilon \to 0^+$ that are centred at $\omega = 0$ and $\omega = \omega_R$ respectively;



$\Gamma_2, \Gamma_4, \Gamma_6$ are line segments along the real axis. From (1):

(2) $$0 = \int_{\Gamma_1} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} + \int_{\Gamma_2+\Gamma_4+\Gamma_6} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} + \int_{\Gamma_3} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} + \int_{\Gamma_5} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)}$$

We consider each integral separately.

First integral (3) $$\int_{\Gamma_1} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} = 0,$$ as $R \to \infty$, for the same reason as given after (371) in the notes.

Second integral (4) $$\lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \int_{\Gamma_2+\Gamma_4+\Gamma_6} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} \equiv P\int_{-\infty}^{\infty} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)},$$

where P denotes "Cauchy principal value", as defined in the notes.

Third integral Let $\omega = \varepsilon e^{i\theta}$, $\Rightarrow d\omega/d\theta = i\varepsilon e^{i\theta}$

$$\Rightarrow \lim_{\varepsilon\to 0^+} \int_{\Gamma_3} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} = \lim_{\varepsilon\to 0^+} \int_{\theta=\pi}^{0} \frac{f(\omega)\frac{d\omega}{d\theta}\,d\theta}{\varepsilon e^{i\theta}(\varepsilon e^{i\theta}-\omega_R)}$$

$$= \lim_{\varepsilon\to 0^+} \int_{\theta=\pi}^{0} \frac{f(\varepsilon e^{i\theta})\, i\cancel{\varepsilon e^{i\theta}}\,d\theta}{\cancel{\varepsilon e^{i\theta}}(\varepsilon e^{i\theta}-\omega_R)}$$

(Note that one can do this, and the next, integral by using the "residue theorem".)

6

$$⑤ = \frac{i f(0)}{-\omega_R} \int_{\theta=\pi}^{0} d\theta = \frac{i f(0)}{-\omega_R} \times (-\pi) = \frac{i\pi f(0)}{\omega_R}$$

<u>Fourth integral</u> let $\omega = \omega_R + \varepsilon e^{i\theta} \Rightarrow \frac{d\omega}{d\theta} = i\varepsilon e^{i\theta}$

$$\Rightarrow \int_{\Gamma_5} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} = \lim_{\varepsilon\to 0^+} \int_{\theta=\pi}^{0} \frac{f(\omega_R+\varepsilon e^{i\theta}) \frac{d\omega}{d\theta}\, d\theta}{(\omega_R+\varepsilon e^{i\theta})\varepsilon e^{i\theta}}$$

$$= \lim_{\varepsilon\to 0^+} \int_{\theta=\pi}^{0} \frac{f(\omega_R+\varepsilon e^{i\theta})\, i \cancel{\varepsilon e^{i\theta}}\, d\theta}{(\omega_R+\varepsilon e^{i\theta}) \cancel{\varepsilon e^{i\theta}}}$$

$$⑥ \qquad = \frac{i f(\omega_R)}{\omega_R} \int_{\theta=\pi}^{0} d\theta = -\frac{i\pi f(\omega_R)}{\omega_R}$$

Substitute ③, ④, ⑤, ⑥ into ②, so that:

$$0 = P\int_{-\infty}^{\infty} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)} + \frac{i\pi f(0)}{\omega_R} - \frac{i\pi f(\omega_R)}{\omega_R}.$$

$\to$ 0, since the question states that $f(0)=0$.

$$\Rightarrow \frac{i\pi f(\omega_R)}{\omega_R} = P\int_{-\infty}^{\infty} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)}$$

$$f(\omega_R) = \frac{\omega_R}{i\pi} P\int_{-\infty}^{\infty} \frac{f(\omega)\,d\omega}{\omega(\omega-\omega_R)}$$

$$Re\{f(\omega_R)\} = \frac{\omega_R}{\pi} P\int_{-\infty}^{\infty} \frac{\overbrace{Re\{f(\omega)/i\}}^{Im\{f(\omega)\}}\,d\omega}{\omega(\omega-\omega_R)}$$

$$Re\{f(\omega_R)\} = \frac{\omega_R}{\pi} P\int_{-\infty}^{\infty} \frac{Im\{f(\omega)\}\,d\omega}{\omega(\omega-\omega_R)}$$

Now let:
$\omega \to \omega'$
$\omega_R \to \omega$

$$\Rightarrow Re\{f(\omega)\} = \frac{\omega}{\pi} P\int_{-\infty}^{\infty} \frac{Im\{f(\omega')\}\,d\omega'}{\omega'(\omega'-\omega)} \qquad \text{QED.}$$

Problem 57 Apply the operator $(\nabla^2 + k^2)$ to (380):

$$(\nabla^2+k^2)\left\{-\frac{1}{4\pi}\iiint G(\vec{x},\vec{x}')\,U(\vec{x}')\,\psi(\vec{x}')\,dx'dy'dz'\right\}$$

$$= -\frac{1}{4\pi}\iiint \Big[\underbrace{(\nabla^2+k^2)G(\vec{x},\vec{x}')}_{(379)}\Big]U(\vec{x}')\psi(\vec{x}')\,dx'dy'dz'$$

$$= -\frac{1}{4\pi}\iiint\left[-4\pi\,\delta(\vec{x}-\vec{x}')\right]U(\vec{x}')\psi(\vec{x}')\,dx'dy'dz'$$

$$= \iiint \delta(\vec{x}-\vec{x}')\,U(\vec{x}')\,\psi(\vec{x}')\,dx'dy'dz'$$

$= U(\vec{x})\,\psi(\vec{x})$. Hence the quantity in braces, in the first line of this equation, is a "solution" to (378) in the notes.

We put 'solution' in inverted commas since it's not a solution at all → we end up recasting the differential equation (378) as the integral equation (381).

Problem 58 Take (388), and replace the cartesian coordinates $\vec{k}' \equiv (k_x', k_y', k_z')$ with the spherical polar coordinates $\vec{k}' \equiv (k', \theta', \phi')$. Hence (388) becomes:

$$① \quad G(\vec{x}) = \frac{1}{2\pi^2}\int_{k'=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} k'^2 \sin\theta'\,dk'\,d\theta'\,d\phi' \frac{\times \exp(i\vec{k}'\cdot\vec{x})}{k'^2-k^2}$$

~~NOW, we are free to orient our spherical polar coordinates however we like ... so let us orient them so $\vec{k}'$ is parallel to $\vec{x}$ ... therefore~~

$$\cancel{② \quad \vec{k}'\cdot\vec{x} = |\vec{k}'|\underbrace{|\vec{x}|}_{r}\cos(0) = k'r}$$

~~, so ① becomes:~~

$$\cancel{③ \quad G(\vec{x}) = G(|\vec{x}|) \equiv G(r) = \frac{1}{2\pi^2}\int_{k'=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} \frac{k'^2\sin\theta'\,dk'\,d\theta'\,d\phi'\,e^{ik'r}}{k'^2-k^2}}$$

$$= \frac{1}{2\pi^2}\int_{k'=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} \frac{k'^2 \sin\theta' dk' d\theta' d\phi' \exp(ik'r\cos\theta')}{k'^2-k^2}$$

(now do the ϕ' integral, and re-order remaining integrals)

$$= \frac{2\pi}{2\pi^2}\int_{k'=0}^{\infty}\frac{k'^2 dk'}{k'^2-k^2}\left[\underbrace{\int_{\theta'=0}^{\pi} \sin\theta' \exp(ik'r\cos\theta')d\theta'}\right]$$

$$\left(\text{Let } \lambda \equiv \cos\theta' \Rightarrow \frac{d\lambda}{d\theta'} = -\sin\theta' \Rightarrow \sin\theta' d\theta' = -d\lambda\right)$$

$$= \frac{1}{\pi}\int_{k'=0}^{\infty}\frac{k'^2 dk'}{k'^2-k^2}\left[\int_{\lambda=1}^{\lambda=-1}(-d\lambda)\exp(ik'r\lambda)\right]$$

$$= \frac{1}{\pi}\int_{k'=0}^{\infty}\frac{k'^2 dk'}{k'^2-k^2}\left[\int_{\lambda=-1}^{1} d\lambda \exp(ik'r\lambda)\right]$$ Now perform the integral...

$$= \frac{1}{\pi}\int_{k'=0}^{\infty}\frac{k'^2 dk'}{k'^2-k^2}\left\{\frac{e^{ik'r\lambda}}{ik'r}\right\}_{\lambda=-1}^{\lambda=+1}$$

$$= \frac{1}{\pi}\int_{k'=0}^{\infty}\frac{k'^2 dk'}{k'^2-k^2}\times\frac{1}{ik'r}\times\left(e^{ik'r}-e^{-ik'r}\right)$$ $$\left(\text{Note: } \sin(k'r) = \frac{e^{ik'r}-e^{-ik'r}}{2i}\right)$$

$$= \frac{1}{\pi}\int_{k'=0}^{\infty}\frac{k'^2 dk'}{k'^2-k^2}\times\frac{2}{k'r}\times\sin(k'r)$$

$$= \frac{2}{\pi r}\int_{k'=0}^{\infty}\frac{k' dk'}{k'^2-k^2}\sin(k'r)$$

$$= \frac{-2}{\pi r}\frac{d}{dr}\int_{k'=0}^{\infty} dk' \underbrace{\frac{1}{k'^2-k^2}\cos(k'r)}$$ $$\left(\text{Note: } k'\sin(k'r) = -\frac{d}{dr}\cos(k'r)\right)$$

(integral is an even function of k'... therefore we may extend the lower limit to $-\infty$, and divide by 2)

$$= \frac{-1}{\pi r}\frac{d}{dr}\int_{k'=-\infty}^{\infty}\frac{\cos(k'r)dk'}{k'^2-k^2}$$

$$= \frac{-1}{\pi r}\frac{d}{dr}\int_{-\infty}^{\infty}\frac{dk'[\cos(k'r)+\underbrace{i\sin(k'r)}]}{k'^2-k^2}$$

(we can "add" this piece, because it integrates to zero, as $\sin(k'r)/(k'^2-k^2)$ is an odd function of k')

$$= \frac{-1}{\pi r}\frac{d}{dr}\int_{-\infty}^{\infty}\frac{dk' \exp(ik'r)}{k'^2-k^2}$$

This is the required result!

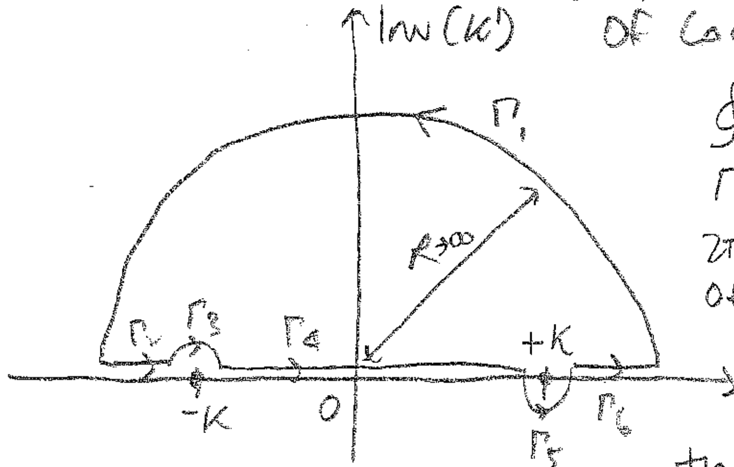
PS: If you can find a more elegant way to do this, please let me know!

10/4/2004

Problem 59 From (389), $G(\vec{x}) = \frac{-1}{\pi r}\frac{d}{dr}\int_{-\infty}^{\infty}\frac{\exp(ik'r)\,dk'}{k'^2-k^2}$. ①

We will now evaluate the integral:

② $\int_{-\infty}^{\infty}\frac{\exp(ik'r)\,dk'}{k'^2-k^2} = \int_{-\infty}^{\infty}\frac{\exp(ik'r)\,dk'}{(k'-k)(k'+k)}$, using the residue theorem of complex analysis. By the residue theorem of complex analysis, the integral:



$\oint_\Gamma \frac{\exp(ik'r)\,dk'}{(k'-k)(k'+k)}$ is equal to $2\pi i$ multiplied by the residue of all poles enclosed within the contour $\Gamma \equiv \Gamma_1 + \Gamma_2 + \ldots + \Gamma_6$. Note that Γ_1 is a semicircle of radius R ... R will be taken to ∞. Also, Γ_3 and Γ_5 are semicircles of vanishingly small radius. Now, the residue at $k'=k$ is: $\lim_{k'\to k}\frac{\cancel{(k'-k)}\exp(ik'r)}{\cancel{(k'-k)}(k'+k)} = \frac{\exp(ikr)}{2k}$, hence:

$$\oint_\Gamma \frac{\exp(ik'r)\,dk'}{(k'-k)(k'+k)} = 2\pi i \times \frac{\exp(ikr)}{2k} = \frac{\pi i}{k}\exp(ikr).$$

Since the contribution, to the integral, of the integral over the contour Γ_1 will vanish (sorry for the bad grammar!), we have:

$$\int_{-\infty}^{+\infty}\frac{\exp(ik'r)\,dk'}{(k'-k)(k'+k)} = \frac{\pi i}{k}\exp(ikr),$$ so that ① becomes:

$$G(\vec{x}) = \frac{-1}{\pi r}\frac{d}{dr}\frac{\pi i}{k}\exp(ikr)$$

$r \equiv |\vec{x}|$

$$= \frac{-i}{kr}\frac{d}{dr}\exp(ikr) = \frac{-i}{kr}\times ik\exp(ikr) = \frac{1}{r}\exp(ikr)$$

Hence $G(\vec{x}) = \frac{\exp(ik|\vec{x}|)}{|\vec{x}|}$ Q.E.D. Note: there is evidently an ambiguity in the choice of contour above. If we had chosen [contour] instead of [contour], we would have obtain a different Green function, which would not satisfy "outgoing" boundary conditions at infinity.

11/4/2004

Problem 60

Before we do anything else, let us get equation (396) into an alternative form, which is more useful for many problems. Merge the two complex exponentials together, to give:

① $\exp[-i(\underbrace{k\hat{x}}_{\text{this is the scattered wave vector, } \vec{k}_{scattered}}\cdot\vec{x}' - \underbrace{kz}_{\text{this is } \vec{k}_{incident}\cdot\vec{x}'})]$

this is $\vec{k}_{incident}\cdot\vec{x}'$, where $\vec{k}_{incident}$ is the incident wave-vector

$$= \exp[-i(\vec{k}_{scattered} - \vec{k}_{incident})\cdot\vec{x}']$$

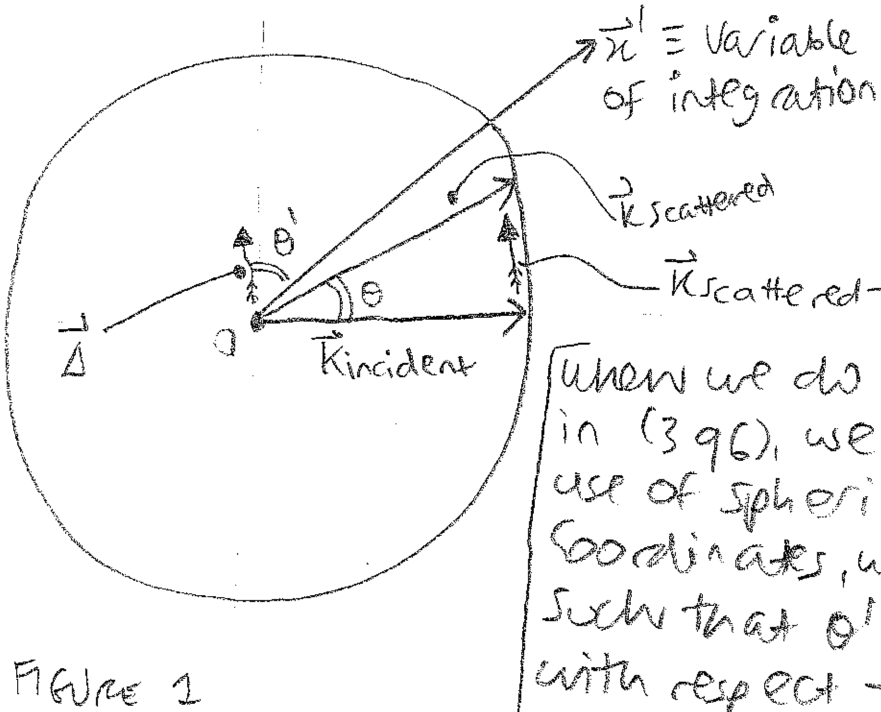


FIGURE 1

{Note that θ, which will appear in the final result, is a "scattering angle".}

[When we do the integration in (396), we shall make use of spherical polar coordinates, which are such that θ' is the angle with respect to the vector $\vec{\Delta}$, when the tail of $\vec{\Delta}$ is placed at the origin of coordinates o.]

Making use of ① above, we see that (396) in the notes becomes:

② $$f(\hat{x}) = \frac{-1}{4\pi}\iiint dx'dy'dz' \exp[-i\vec{\Delta}\cdot\vec{x}']\, U(\vec{x}')$$

Now we transform to spherical polar coordinates, noting the statement in square brackets above:

③ $$f(\hat{x}) = \frac{-1}{4\pi}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\int_{r'=0}^{\infty} r'^2 \sin\theta' \, dr' d\theta' d\phi' \, U(r',\theta',\phi')\, e^{-i\vec{\Delta}\cdot\vec{x}'}$$

Now, bearing FIGURE 1 in mind, we have:

(4) $\vec{\Delta}\cdot\vec{x}' = |\vec{\Delta}||\vec{x}'|\cos\theta'$

$= |\vec{K}_{scattered} - \vec{K}_{incident}|\, r'\cos\theta'$

$= \sqrt{(\vec{K}_{scattered} - \vec{K}_{incident})\cdot(\vec{K}_{scattered} - \vec{K}_{incident})}\; r'\cos\theta'$

$= \sqrt{\underbrace{|\vec{K}_{scattered}|^2}_{k^2} - 2\vec{K}_{scattered}\cdot\vec{K}_{incident} + \underbrace{|\vec{K}_{incident}|^2}_{k^2}}\; r'\cos\theta'$

$= \sqrt{2k^2 - 2|\vec{K}_{scattered}||\vec{K}_{incident}|\cos\theta}\; r'\cos\theta'$ ← see FIG. 1

$= \sqrt{2k^2 - 2k^2\cos\theta}\; r'\cos\theta'$

$= \sqrt{2k^2(1-\cos\theta)}\; r'\cos\theta'$

$= \sqrt{2k^2 \times 2\sin^2\frac{\theta}{2}}\; r'\cos\theta'$

$= 2k\sin\left(\frac{\theta}{2}\right) r'\cos\theta'$

$$\begin{aligned}
\sin^2\theta &= \frac{1}{(2i)^2}(e^{i\theta}-e^{-i\theta})(e^{i\theta}-e^{-i\theta}) \\
&= \frac{-1}{4}(e^{2i\theta}-2+e^{-2i\theta}) \\
&= \frac{1}{2}(1-\cos 2\theta) \\
\Rightarrow \sin^2\frac{\theta}{2} &= \frac{1}{2}(1-\cos\theta) \\
\Rightarrow 2\sin^2\frac{\theta}{2} &= 1-\cos\theta
\end{aligned}$$

Also, we now assume that $U(\vec{x}')$ is a central potential, i.e. that it depends only on r', so that $U(r',\theta',\phi') \to U(r')$. Together with (4), this implies that (3) becomes:

(5) $$f(\hat{x}) = \frac{-1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi'=0}^{2\pi}\int_{r'=0}^{\infty} r'^2 \sin\theta'\, dr' d\theta' d\phi'\, U(r')\, e^{-2ik\sin\left(\frac{\theta}{2}\right) r'\cos\theta'}$$

Now do the ϕ' integral, so that:

(6) $$f(\hat{x}) = \frac{-1}{2}\int_{r'=0}^{\infty} dr'\, r'^2\, U(r') \int_{\theta=0}^{\pi} d\theta' \sin\theta'\, e^{-2ik\sin\left(\frac{\theta}{2}\right) r'\cos\theta'}$$

We do the θ' integral first. Let $\lambda = \cos\theta' \Rightarrow d\lambda/d\theta' = -\sin\theta'$
$\Rightarrow \sin\theta' d\theta' = -d\lambda$

(7) $$f(\hat{x}) = \frac{-1}{2}\int_{r'=0}^{\infty} dr'\, r'^2\, U(r') \int_{\lambda=1}^{-1} (-d\lambda)\, e^{-2ik\sin\left(\frac{\theta}{2}\right)\lambda r'}$$

Now let $\Delta \equiv 2k\sin\left(\frac{\theta}{2}\right)$ (8), so that (7) becomes:

$$⑨\ f(\hat{x}) = -\frac{1}{2}\int_{r'=0}^{\infty} dr' r'^2 u(r') \int_{\lambda=-1}^{1} d\lambda\, e^{-i\Delta\lambda r'}$$

$$= -\frac{1}{2}\int_{r'=0}^{\infty} dr' r'^2 u(r') \left[\frac{e^{-i\Delta\lambda r'}}{-i\Delta r'}\right]_{\lambda=-1}^{\lambda=+1}$$

$$= -\Delta^{-1}\int_{r'=0}^{\infty} dr' r' u(r') \sin(\Delta r')$$

We are now ready to make use of the particular central potential required by this problem. From the line of text immediately after equation (378), we can convert the potential $V = V_0 \exp(-\alpha r)/(\alpha r)$ into the scaled potential u via $u(r) = 2mV/\hbar^2$. Hence we have:

$$⑩\ u(r') = \frac{2mV_0 \exp(-\alpha r')}{\hbar^2 \alpha r'},$$

which may be substituted into ⑨ to give:

$$⑪\ f(\hat{x}) = -\Delta^{-1}\int_{r'=0}^{\infty} dr' \cancel{r'} \frac{2mV_0}{\hbar^2 \alpha \cancel{r'}} e^{-\alpha r'} \sin(\Delta r')$$

$$= \frac{-2mV_0}{\hbar^2 \alpha \Delta}\int_0^{\infty} dr' e^{-\alpha r'} \sin(\Delta r')$$

write as $\frac{1}{2i}(e^{i\Delta r'} - e^{-i\Delta r'})$

... multiply together exponentials ... the resulting integral is straightforward

$$= \frac{-2mV_0}{\hbar^2 \alpha \Delta} \times \frac{\Delta}{\alpha^2 + \Delta^2}$$

$$= \frac{-2mV_0}{\hbar^2 \alpha} \times \frac{1}{\alpha^2 + \Delta^2} \overset{⑧}{=} \frac{-2mV_0}{\hbar^2 \alpha} \times \frac{1}{\alpha^2 + 4k^2 \sin^2 \frac{\theta}{2}}$$

The associated differential cross section, $d\sigma/d\omega$, is the square modulus of this:

$$⑫\ \frac{d\sigma}{d\omega} = \frac{4m^2 V_0^2}{\hbar^4 \alpha^2} \times \frac{1}{(\alpha^2 + 4k^2 \sin^2 \frac{\theta}{2})^2}$$

To obtain the cross-section for scattering from a Coulomb potential, let $\alpha \to 0$, $V_0/\alpha \to q_1 q_2$, where q_1 and q_2 are the charges of the things that are scattering. Hence ⑫ becomes:

$$⑬\ \frac{d\sigma}{d\omega} \to \frac{4m^2}{\hbar^4} \times (q_1 q_2)^2 \times \frac{1}{16k^4 \sin^4\left(\frac{\theta}{2}\right)}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{Coulomb}\\\text{scattering}}} = \frac{m^2 q_1^2 q_2^2}{4\underbrace{(\hbar k)}_{\text{momentum, } \vec{p}}{}^4 \sin^4\left(\frac{\theta}{2}\right)}$$

$$= \frac{m^2 q_1^2 q_2^2}{4\,|\vec{p}|^4 \sin^4\left(\frac{\theta}{2}\right)}$$

$$= \frac{1}{16} \times \left(\frac{2m}{\underbrace{|\vec{p}|^2}_{\text{energy } E}}\right)^2 \times \frac{q_1^2 q_2^2}{\sin^4\left(\frac{\theta}{2}\right)}$$

$$= \frac{q_1^2 q_2^2}{16 E^2 \sin^4\left(\frac{\theta}{2}\right)}$$

This is the famous Rutherford cross section, for scattering from a Coulomb potential. Curiously, our calculation (based on the first Born approximation), is the same• as both:

* the exact quantum-mechanical result
* the classical result.

This is a famous peculiarity of the Coulomb potential. Note, also, that the Rutherford formula is of great historical significance, since it was used in the famous interpretation of Rutherford's experiments on α-particle scattering by gold foils.

• i.e., "it gives the same answer as"